

# GENERALIZED AXIALLY SYMMETRIC POTENTIALS WITH DISTRIBUTIONAL BOUNDARY VALUES

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**ABSTRACT.** We study a counterpart of the classical Poisson integral for a family of weighted Laplace differential equations in Euclidean half space, solutions of which are known as generalized axially symmetric potentials. These potentials appear naturally in the study of hyperbolic Brownian motion with drift. We determine the optimal class of tempered distributions which by means of the so-called  $\mathcal{S}'$ -convolution can be extended to generalized axially symmetric potentials. In the process, the associated Dirichlet boundary value problem is solved, and we obtain sharp order relations for the asymptotic growth of these extensions.

## 1. INTRODUCTION

Consider in  $n + 1$  dimensions the elliptic partial differential equation

$$(1.1) \quad D_\alpha u \equiv y^{-\alpha} \left( \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\alpha}{y} \frac{\partial u}{\partial y} \right) = 0,$$

where  $\alpha$  is an arbitrary real parameter. When  $\alpha = 0$  we recover the classical Laplace equation, and when  $\alpha$  is a negative integer and  $n = 1$  then (1.1) is satisfied by the family of axially symmetric harmonic functions in  $(2 - \alpha)$ -dimensional space, considered in a meridian plane. Solutions to (1.1) have therefore historically been referred to as generalized axially symmetric potentials, see the exposition by Weinstein [30]. This theory proved to be a very strong tool allowing treatment of various problems in for example fluid mechanics and generalized Tricomi equations [30, 31]. In this context, the operator  $y^{\alpha+1} D_\alpha$  has traditionally been denoted by  $L_k$  with parameter  $k = -\alpha$ , that is,

$$L_k u \equiv y \left( \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial y^2} \right) + k \frac{\partial u}{\partial y}, \quad k \in \mathbb{R}.$$

The equation  $D_\alpha u = 0$  is the Laplace-Beltrami equation in the Riemannian space defined by the metric

$$ds^2 = y^{2\alpha/(1-n)} \left( \sum_{i=1}^n dx_i^2 + dy^2 \right), \quad n > 1.$$

This fact has recently led to the appearance of the operator  $D_\alpha$  in connection with the study of so-called  $(n + 1)$ -dimensional hyperbolic Brownian motion with drift. This area has been of much interest lately, since it is related to geometric Brownian motion and Bessel processes and has applications to risk theory, see the recent

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work by Małeck and Serafin [22] and the references therein. Contributions have also been made by among others Baldi, Byczkowski, Casadio Tarabusi, Figà-Talamanca, Graczyk, Ryznar, Stós, and Yor [5, 6, 9, 10]. We recall that if  $\mathbb{H}^{n+1}$  denotes the half space model of  $(n+1)$ -dimensional real hyperbolic space, that is,  $\mathbb{H}^{n+1}$  is the space  $\{(x, y) \in \mathbb{R}^{n+1} : y > 0\}$  endowed with the metric  $ds^2 = y^{-2}(dx_1^2 + \cdots + dx_n^2 + dy^2)$ , then an  $(n+1)$ -dimensional hyperbolic Brownian motion with drift in  $\mathbb{H}^{n+1}$  is defined as a diffusion corresponding to the system of stochastic differential equations

$$(1.2) \quad \begin{cases} dX_t = Y_t dW_t, \\ dY_t = Y_t dB_t - (\mu - \frac{1}{2})Y_t dt. \end{cases}$$

Here  $W_t$  and  $B_t$  are independent  $n$ -dimensional and one-dimensional Brownian motions, respectively. The generator of this diffusion is given by

$$y^2 \left( \sum_{i=1}^n \partial_{x_i}^2 + \partial_y^2 \right) - (2\mu - 1)y\partial_y$$

divided by a factor 2, which is a non-constant multiple of  $D_\alpha$  for the parameter value  $\alpha = 2\mu - 1$ . The case  $\mu = n/2$  corresponds to classical hyperbolic Brownian motion on  $\mathbb{H}^{n+1}$ .

In this paper we shall study a half space boundary value problem for the operator  $D_\alpha$ . Let  $\mathbb{R}_+^{n+1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$  denote the half space in  $n+1$  dimensions. We will identify the boundary of  $\mathbb{R}_+^{n+1}$  with  $\mathbb{R}^n$ . Introduce the weight function  $\gamma_\alpha(x, y) = y^{-\alpha}$  for  $(x, y) \in \mathbb{R}_+^{n+1}$ , and note that  $D_\alpha$  can be written in divergence form

$$D_\alpha u = \operatorname{div}(\gamma_\alpha \nabla u), \quad \gamma_\alpha(x, y) = y^{-\alpha}, \quad (x, y) \in \mathbb{R}_+^{n+1}.$$

Since the behavior of solutions to  $D_\alpha u = 0$  are essentially different for the two parameter ranges  $\alpha > -1$  and  $\alpha \leq -1$  (see the result due to Huber [20], included below as Theorem 2.1) we shall restrict our attention to the parameter range  $\alpha > -1$  and study the Dirichlet problem

$$(1.3) \quad \begin{cases} D_\alpha u = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ u = f & \text{in } \mathbb{R}^n. \end{cases}$$

We will assume that the boundary data  $f \in \mathcal{S}'$  is a tempered distribution in  $\mathbb{R}^n$ ; the boundary condition is to be understood as  $\lim_{y \rightarrow 0} u_y = f$  in  $\mathcal{S}'$ , where

$$(1.4) \quad u_y(x) = u(x, y), \quad x \in \mathbb{R}^n,$$

for  $y > 0$ .

Due to the definition of the weight  $\gamma_\alpha$ , (1.3) is singular on the hyperplane  $y = 0$ . We mention here that  $D_\alpha$  is related to a certain weighted complex Laplace operator  $\Delta_\alpha$  in the unit disc  $\mathbb{D}$ , which exhibits similar behavior near the boundary of  $\mathbb{D}$ . In fact, when  $n = 1$  the operator  $D_\alpha$  is realized as (a multiple of) the symmetric part of the differential operator

$$u(z) \mapsto \partial_z \gamma_\alpha(y) \bar{\partial}_z u(z), \quad z = x + iy \in \mathbb{C}, \quad y > 0,$$

where  $\partial$  and  $\bar{\partial}$  are the usual complex derivatives. The mentioned weighted Laplace operator  $\Delta_\alpha$  is obtained by replacing the weight function  $\gamma_\alpha$  by its counterpart in the unit disc, the so-called standard weight  $z \mapsto (1 - |z|^2)^{-\alpha}$  for  $z \in \mathbb{D}$ , appearing in connection to Bergman space theory. The corresponding Dirichlet problem for  $\Delta_\alpha$

in the unit disc was recently solved by the author in collaboration with A. Olofsson [24]. We also mention the recent paper by Olofsson [23] which in a certain sense studies the unit disc analog of the Dirichlet problem (1.3). The family of operators studied by Olofsson [23] has been shown to be connected to weighted integrability of polyharmonic functions in the unit disc by Borichev and Hedenmalm [8]. See also Hedenmalm [15].

The singular or degenerate behavior of  $D_\alpha$  near the boundary means that the theory for strictly elliptic partial differential equations is not applicable to (1.3). This notwithstanding, much is still known about the existence and uniqueness of solutions to the Dirichlet problem (1.3) when the data is regular. In particular, the notion corresponding to a Poisson integral appears in Weinstein [30] in the case  $n = 1$ , using a kernel function corresponding to

$$\mathcal{K}_\alpha(x, y) = \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2)\pi^{n/2}} \cdot \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+n+1)/2}}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

where  $\Gamma(s)$  is the Gamma function. We study properties of this kernel in Section 2. (In the context of hyperbolic Brownian motion, the term *probability density function* is commonly used.) Moreover, a fundamental solution in any dimension was found by Diaz and Weinstein [13], and a generalized Poisson kernel for the Dirichlet problem in a hemisphere was also provided by Huber [20]. Similar formulas appear in the more recent paper by Caffarelli and Silvestre [11] (and in other subsequent work) on the fractional Laplacian  $(-\Delta)^\mu$ ,  $0 < \mu < 1$ , where  $(-\Delta)^\mu$  was shown to be related to the extension problem (1.3) for  $\alpha = 2\mu - 1$  through the Dirichlet to Neumann map.

However, for boundary data  $f \in \mathcal{S}'$ , it is not immediately clear how to define the “Poisson integral of  $f$ ” by means of the kernel function  $\mathcal{K}_\alpha$ . The natural choice would be through convolution  $\mathcal{K}_{\alpha,y} * f$  in the sense of distributions, with  $\mathcal{K}_{\alpha,y}$  interpreted in accordance with (1.4), but this is not applicable in our case since the Fourier transform of  $\mathcal{K}_{\alpha,y}$  is not smooth at the origin, see Theorem 2.6. (Recall that the convolution  $u * v$  was originally defined by Schwartz [25] for pairs  $u \in \mathcal{O}'_C$  and  $v \in \mathcal{S}'$ , where  $\mathcal{O}'_C$  is the space of rapidly decaying distributions, and that the Fourier transform is an isomorphism between  $\mathcal{O}'_C$  and the space  $\mathcal{O}_M$  of slowly growing smooth functions, see Schwartz [25, Chapitre VII].) To circumvent this problem we shall use the so-called  $\mathcal{S}'$ -convolution proposed by Hirata and Ogata [16] and later given an equivalent form by Shiraishi [26].

In Section 3 we recall the definitions of certain weighted spaces of distributions (continuously embedded in  $\mathcal{S}'$ ), and determine the optimal class of tempered distributions  $f$  for which the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha,y} * f$  is well defined for all  $y > 0$ , see Theorem 3.3. In Section 4 we define the Poisson integral  $\mathcal{K}_\alpha[f] : (x, y) \mapsto \mathcal{K}_{\alpha,y} * f(x)$  for  $f$  in this class, and show that it has boundary limit  $f$  in  $\mathcal{S}'$ , see Theorem 4.3. We also establish that  $u = \mathcal{K}_\alpha[f]$  solves  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$  for such  $f$ , thus proving existence of solutions to the Dirichlet problem (1.3), see Corollary 4.4. A similar approach has been used by Alvarez, Guzmán-Partida and Skórník [2] to characterize the tempered distributions that are  $\mathcal{S}'$ -convolvable with the classical Poisson kernel  $\mathcal{K}_0$  for the half space, and further used by Alvarez, Guzmán-Partida and Pérez-Estevea [1] to study harmonic extensions of distributions. To prove our results we adapt the ideas found in the mentioned papers to the full parameter range  $\alpha > -1$ . At the end of Section 4 we also calculate the kernel function for the Dirichlet problem  $D_\alpha u = 0$  in the half space  $y > \eta$  where  $\eta > 0$ , see Proposition

4.5. This kernel is the density of the *hitting distribution* appearing in hyperbolic Brownian motion.

In Section 5 we study asymptotic behavior of the Poisson integral  $\mathcal{K}_\alpha[f]$ . By using methods similar to those used by Siegel and Talvila [27] to study the classical Poisson integral in the harmonic case  $\alpha = 0$ , the order relations for the asymptotic growth that we obtain (Theorem 5.2) are shown to be sharp. For comparison, we also consider the issue of uniqueness of solutions: As evidenced by the function  $u(x, y) = y^{\alpha+1}$ , which solves  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$  and vanishes at the boundary if  $\alpha > -1$ , solutions to (1.3) are not unique in general unless additional restrictions of growth at infinity are imposed. We include one result in this direction, proved by using a Phragmén-Lindelöf principle due to Huber [19] together with a regularization argument, see Corollary 5.4. However, the growth conditions imposed are not compatible with the established asymptotic behavior of the Poisson integral  $\mathcal{K}_\alpha[f]$  for general  $f$ , so this does unfortunately not lead to a satisfactory representation theory.

We finally wish to mention that the operator  $D_\alpha$  superficially resembles the governing operator in what is known as Calderón's inverse conductivity problem; however, the conditions on the weight function are totally different. In fact, let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  for  $n \geq 2$ , and let  $\gamma$  be a real-valued function in  $L^\infty(\mathbb{R}^n)$  with a positive lower bound. Consider the conductivity equation

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

In 1980, Calderón [12] posed the question whether the conductivity  $\gamma$  could be recovered from the boundary measurements as described by the Dirichlet to Neumann map  $\Lambda_\gamma$ . This problem, known in medical imaging as Electrical Impedance Tomography, has been intensely studied and numerous positive results are known under slightly stronger regularity assumptions on  $\gamma$ . In 2 dimensions, the problem was recently solved by Astala and Päivärinta [4] who showed that  $\gamma \in L^\infty(\mathbb{R}^2)$  is completely determined by  $\Lambda_\gamma$  even if the boundedness assumption on the domain  $\Omega$  is dropped. For more on this, we refer to the mentioned paper and the references therein.

## 2. THE KERNEL FUNCTION

In this section we discuss the kernel function  $\mathcal{K}_\alpha$  for the Dirichlet problem (1.3) mentioned in the introduction, and calculate its Fourier transform. However, we first indicate the difference in behavior of solutions to  $D_\alpha u = 0$  in the two parameter ranges  $\alpha > -1$  and  $\alpha \leq -1$ . The following result is due to Huber [20], stated here using our choice of notation but otherwise unchanged.

**Theorem 2.1** (A. Huber). *Let  $u$  be a solution of  $D_\alpha u = 0$ , defined in a region  $G$ , the boundary of which contains an open subset  $S$  of  $\{(x, y) \in \mathbb{R}^{n+1} : y = 0\}$ . If  $u$  assumes the boundary value 0 on  $S$ , then we may conclude*

- (a) *for  $\alpha \leq -1$  :  $u \equiv 0$  throughout  $G$ ,*
- (b) *for  $\alpha > -1$  :  $u$  can be represented in the form  $u = y^{\alpha+1}v(x, y)$ , where  $v$  is analytic on  $G \cup S$  and satisfies  $D_{-(2+\alpha)}v = 0$ . Conversely each function of this type fulfills the above hypotheses.*

In particular, this result implies that a Green's function for  $\mathbb{R}_+^{n+1}$  does not exist when  $\alpha \leq -1$ , while for  $\alpha > -1$  Green's function is known, see Weinstein [30] and Diaz and Weinstein [13] for the case  $n = 1$  and  $n \geq 2$ , respectively. We shall therefore henceforth restrict our attention to the parameter range  $\alpha > -1$ . Note also that Theorem 2.1 can be viewed as a uniqueness result since it indicates how far a solution is determined by its boundary values. We shall return briefly to the question of uniqueness for the Dirichlet problem (1.3) at the end of Section 5.

In what follows, we will permit us to write  $x^2$  to denote  $x_1^2 + \cdots + x_n^2$  whenever  $x \in \mathbb{R}^n$ . We will also assume that all function spaces under consideration below are defined on  $\mathbb{R}^n$  unless explicitly stated otherwise.

**Definition 2.2.** Let  $\alpha > -1$ . Define the kernel  $\mathcal{K}_\alpha$  by

$$\mathcal{K}_\alpha(x, y) = \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2)\pi^{n/2}} \cdot \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+n+1)/2}}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

where  $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$  for  $s > 0$  is the Gamma function.

Note that the classical Poisson kernel for the half space  $\mathbb{R}_+^{n+1}$ ,

$$P(x, y) = \mathcal{K}_0(x, y) = \frac{\Gamma((n + 1)/2)}{\pi^{(n+1)/2}} \cdot \frac{y}{(x^2 + y^2)^{(n+1)/2}}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

is obtained for the parameter value  $\alpha = 0$ . Note also that when  $\alpha = n - 1$  we recover the Poisson kernel for the half space model  $\mathbb{H}^{n+1} = \mathbb{H}^{\alpha+2}$  of real hyperbolic space

$$P_{\mathbb{H}^{n+1}}(x, y) = \mathcal{K}_{n-1}(x, y) = \frac{\Gamma(n)}{\Gamma(n/2)\pi^{n/2}} \cdot \frac{y^n}{(x^2 + y^2)^n}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

see Guivarc'h, Ji, and Taylor [14]. Similarly, the family of kernels  $\mathcal{K}_\alpha$  are naturally related to the differential operators  $D_\alpha$  for  $\alpha > -1$ , see Theorem 2.4 below. In preparation for the proof, we calculate the  $L^1$  norm of the function  $\mathcal{K}_{\alpha,y}$ .

**Lemma 2.3.** Let  $\alpha > -1$ . Let  $\mathcal{K}_\alpha$  be given by Definition 2.2. Then

$$\|\mathcal{K}_{\alpha,y}\|_{L^1} = \int \mathcal{K}_{\alpha,y}(x) dx = 1,$$

where  $\mathcal{K}_{\alpha,y}$  is defined in accordance with (1.4).

*Proof.* Define the auxiliary function  $u$  by

$$u(x, y) = \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+n+1)/2}}, \quad (x, y) \in \mathbb{R}_+^{n+1}.$$

In view of Definition 2.2, the theorem follows if we show that  $u$  satisfies

$$\|u_y\|_{L^1} = \frac{\Gamma((\alpha + 1)/2)\pi^{n/2}}{\Gamma((\alpha + n + 1)/2)}, \quad y > 0,$$

where  $u_y$  is defined in accordance with (1.4). To this end, we first note that for each  $y > 0$  we have  $u_y(x) > 0$  for all  $x \in \mathbb{R}^n$ , and that a change of variables  $x/y \mapsto x$  shows that  $\int u_y(x) dx = \int u_1(x) dx$ . Switching to spherical coordinates, we have

$$\|u_1\|_{L^1} = \int \frac{dx}{(1 + x^2)^{(\alpha+n+1)/2}} = \omega_{n-1} \int_0^\infty \frac{r^{n-1} dr}{(1 + r^2)^{(\alpha+n+1)/2}},$$

where  $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$  is the surface area of the unit sphere  $S^{n-1}$ . The change of variables  $r = \sqrt{1/t - 1}$  and a straightforward computation shows that

$$\begin{aligned} 2 \int_0^\infty \frac{r^{n-1} dr}{(1+r^2)^{(\alpha+n+1)/2}} &= \int_0^1 t^{(\alpha+1)/2-1} (1-t)^{n/2-1} \\ &= B((\alpha+1)/2, n/2) = \frac{\Gamma((\alpha+1)/2)\Gamma(n/2)}{\Gamma((\alpha+n+1)/2)}, \end{aligned}$$

where  $B(x, y)$  is the Beta integral, see Andrews, Askey, and Roy [3, Theorem 1.1.4]. Hence,

$$\|u_1\|_{L^1} = \frac{\pi^{n/2}}{\Gamma(n/2)} \cdot 2 \int_0^\infty \frac{r^{n-1} dr}{(r^2+1)^{(\alpha+n+1)/2}} = \frac{\pi^{n/2}\Gamma((\alpha+1)/2)}{\Gamma((\alpha+n+1)/2)},$$

which completes the proof.  $\square$

**Theorem 2.4.** *Let  $\alpha > -1$ . Then the function  $\mathcal{K}_\alpha$  given by Definition 2.2 is a solution to the equation  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$ , and has the boundary limit  $\lim_{y \rightarrow 0} \mathcal{K}_{\alpha,y} = \delta_0$  in  $\mathcal{S}'$ , where  $\mathcal{K}_{\alpha,y}$  is defined in accordance with (1.4).*

*Proof.* That  $D_\alpha \mathcal{K}_\alpha(x, y) = 0$  for  $(x, y) \in \mathbb{R}_+^{n+1}$  follows by straightforward differentiation. We proceed to analyze the boundary limit of  $\mathcal{K}_\alpha$ . For  $y > 0$  we have

$$(2.1) \quad \mathcal{K}_{\alpha,y}(x) = \mathcal{K}_\alpha(x, y) = y^{-n} \mathcal{K}_{\alpha,1}(x/y), \quad x \in \mathbb{R}^n,$$

a fact that was already used in the proof of Lemma 2.3. By the same lemma, the function  $\mathcal{K}_{\alpha,1}$  satisfies  $\int \mathcal{K}_{\alpha,1}(x) dx = 1$ . A standard construction of approximate identities now ensures that  $\lim_{y \rightarrow 0} \mathcal{K}_{\alpha,y} = \delta_0$  in  $\mathcal{S}'$ , see Katznelson [21, Section VI.1.13] or Hörmander [17, Theorem 1.3.2].  $\square$

By virtue of Definition 2.2 and Theorem 2.4, it is clear that the composition

$$v : (x, y) \mapsto \mathcal{K}_\alpha(rx + t, ry), \quad r > 0, t \in \mathbb{R}^n,$$

also solves the equation  $D_\alpha v = 0$  in  $\mathbb{R}_+^{n+1}$ . This structural property is in fact shared by all solutions to  $D_\alpha u = 0$ .

**Proposition 2.5.** *Let  $\alpha > -1$ , and let  $u$  be a solution to  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$ . Let  $r > 0$  and  $t \in \mathbb{R}^n$  and set  $v(x, y) = u(rx + t, ry)$ . Then  $D_\alpha v = 0$  in  $\mathbb{R}_+^{n+1}$ .*

*Proof.* If  $(x, y) \in \mathbb{R}_+^{n+1}$  then  $(rx + t, ry) \in \mathbb{R}_+^{n+1}$  when  $r > 0$  and  $t \in \mathbb{R}^n$ . Differentiation gives

$$\begin{aligned} D_\alpha v(x, y) &= r^2 y^{-\alpha} \Delta u(rx + t, ry) - r \alpha y^{-\alpha-1} \partial u(rx + t, ry) / \partial y \\ &= r^{\alpha+2} ((ry)^{-\alpha} \Delta u(rx + t, ry) - \alpha (ry)^{-\alpha-1} \partial u(rx + t, ry) / \partial y) \\ &= r^{\alpha+2} D_\alpha u(rx + t, ry) = 0, \end{aligned}$$

which completes the proof.  $\square$

Next we analyze the Fourier transform of  $x \mapsto \mathcal{K}_{\alpha,y}(x)$  for  $\alpha > -1$ , that is, the function

$$(2.2) \quad \xi \mapsto \widehat{\mathcal{K}_{\alpha,y}}(\xi) = \int e^{-i\langle x, \xi \rangle} \mathcal{K}_\alpha(x, y) dx.$$

A formula for this function has been obtained before, and can for example be found in the paper by Baldi, Casadio Tarabusi and Figà-Talamanca [5, Section 4] for  $n = 1$  within the framework of hyperbolic Brownian motion with drift, although

this requires some translation between the choice of notation. (In this context, the terminology *characteristic function* is commonly used.) We therefore prefer to include a direct proof using a different method, and we will indicate the connection afterwards. A similar approach was used by Byczkowski, Graczyk and Stós [9] in the special case  $\alpha = n - 1$  corresponding to classical hyperbolic Brownian motion on  $\mathbb{H}^{n+1}$  (compare with the discussion in the introduction above).

Recall that  $\mathcal{K}_{\alpha,y}$  has bounded  $L^1$  norm (independent of  $y > 0$ ) by Lemma 2.3, so the integral (2.2) is absolutely convergent and  $\widehat{\mathcal{K}_{\alpha,y}}(\xi)$  is a continuous function of  $\xi$ . To analyze how  $\widehat{\mathcal{K}_{\alpha,y}}$  depends on  $y$ , note that (2.1) yields the identity

$$(2.3) \quad \widehat{\mathcal{K}_{\alpha,y}}(\xi) = \int e^{-i\langle x, \xi \rangle} \mathcal{K}_{\alpha,1}(x/y) y^{-n} dx = \int e^{-i\langle x, y\xi \rangle} \mathcal{K}_{\alpha,1}(x) dx = \widehat{\mathcal{K}_{\alpha,1}}(y\xi).$$

Since  $x \mapsto \mathcal{K}_{\alpha,1}(x)$  is a radial function, the Fourier transform  $\eta \mapsto \widehat{\mathcal{K}_{\alpha,1}}(\eta)$  is also a radial function. In fact, let  $f_\alpha : [0, \infty) \rightarrow \mathbb{C}$  be the  $\mathcal{C}^\infty$  function defined by

$$f_\alpha(r) = \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2)\pi^{n/2}} \cdot \frac{1}{(1 + r^2)^{(\alpha + n + 1)/2}}, \quad r \geq 0,$$

so that  $\mathcal{K}_{\alpha,1}(x) = f_\alpha(|x|)$ , and let  $J_\nu$  denote the Bessel function of the first kind of order  $\nu$ ,

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{izt} (1 - t^2)^{\nu - \frac{1}{2}} dt, \quad \operatorname{Re} \nu > -1/2,$$

see for example equation (3) on p. 25 in the treatise by Watson [29]. Then we have  $\widehat{\mathcal{K}_{\alpha,1}}(\eta) = F_\alpha(|\eta|)$  where

$$(2.4) \quad F_\alpha(r) = (2\pi)^{n/2} r^{(2-n)/2} \int_0^\infty f_\alpha(s) s^{n/2} J_{(n-2)/2}(rs) ds,$$

see Stein and Weiss [28, Chapter 4, Theorem 3.3]. Note that their definition of the Fourier transform differs from ours by a scaling factor, which explains the difference in appearance between the formulas. Thus, for  $\alpha > -1$  the Fourier transform of  $x \mapsto \mathcal{K}_{\alpha,y}(x)$  can be written as

$$(2.5) \quad \widehat{\mathcal{K}_{\alpha,y}}(\xi) = \frac{2^{n/2}\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2)} (y|\xi|)^{(2-n)/2} \int_0^\infty \frac{s^{n/2} J_{(n-2)/2}(sy|\xi|)}{(1 + s^2)^{(\alpha + n + 1)/2}} ds.$$

By using the properties of  $\mathcal{K}_\alpha$  given by Theorem 2.4, we may (indirectly) evaluate the integral in (2.5). We thus return to the Fourier transform of  $x \mapsto \mathcal{K}_{\alpha,y}(x)$  given by (2.2). In view of (2.3), it is sufficient to study the case  $y = 1$ . Let therefore  $F_\alpha$  be the radial function defined above such that  $\widehat{\mathcal{K}_{\alpha,1}}(y\xi) = F_\alpha(y|\xi|)$ . Since  $\widehat{\mathcal{K}_{\alpha,1}}$  is continuous and  $\widehat{\mathcal{K}_{\alpha,1}}(\eta) \rightarrow 0$  as  $|\eta| \rightarrow \infty$  by the Riemann Lebesgue lemma, we may identify the map  $(\xi, y) \mapsto \widehat{\mathcal{K}_{\alpha,y}}(\xi) = \widehat{\mathcal{K}_{\alpha,1}}(y\xi) = F_\alpha(y|\xi|)$  with a distribution in  $\mathcal{S}'(\mathbb{R}_+^{n+1})$ . Now  $\mathcal{K}_\alpha$  satisfies  $D_\alpha \mathcal{K}_\alpha = 0$  in  $\mathbb{R}_+^{n+1}$ , so by a Fourier transformation with respect to the  $x$  variables we obtain

$$(2.6) \quad \begin{aligned} 0 &= \mathcal{F}_x((\nabla_x \cdot (y^{-\alpha} \nabla_x) + \partial_y y^{-\alpha} \partial_y) \mathcal{K}_{\alpha,y})(\xi) \\ &= y^{-\alpha} \left( \frac{\partial^2}{\partial y^2} - \frac{\alpha}{y} \frac{\partial}{\partial y} - \xi^2 \right) F_\alpha(y|\xi|), \end{aligned}$$



which we interpret in the distributional sense. However, by Hörmander [17, Theorem 4.4.8] any distribution in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$  solving the differential equation in the right-hand side of (2.6) is a  $\mathcal{C}^2$  function of  $y$  with values in  $\mathcal{D}'$ ; in our case, the values will even be in  $\mathcal{S}'$ . Indeed, for fixed  $y > 0$  we can identify  $\mathcal{K}_{\alpha,y}$  with a tempered distribution in  $\mathcal{S}'$ , so its Fourier transform also belongs to  $\mathcal{S}'$  which proves the claim. If we perform the differentiations in (2.6) we find that

$$\begin{aligned} 0 &= y^{-\alpha} \left( \frac{\partial^2}{\partial y^2} - \frac{\alpha}{y} \frac{\partial}{\partial y} - \xi^2 \right) F_{\alpha}(y|\xi|) \\ &= y^{-\alpha} \xi^2 \left( F_{\alpha}''(y|\xi|) - \frac{\alpha}{y|\xi|} F_{\alpha}'(y|\xi|) - F_{\alpha}(y|\xi|) \right), \end{aligned}$$

so we are led to consider the ordinary differential equation

$$(2.7) \quad v''(t) - \frac{\alpha}{t} v'(t) - v(t) = 0.$$

We therefore digress and recall some well-known facts concerning this equation. For a more thorough discussion we refer the reader to the work by Watson [29] and the references therein.

If  $v$  is a solution to (2.7), set  $u(t) = v(t)t^{-\nu}$  with  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ . It is straightforward to check that  $u$  then solves the equation

$$(2.8) \quad t^2 u''(t) + tu'(t) - (t^2 + \nu^2)u(t) = 0.$$

This implies that the general solution to (2.7) is given by  $t \mapsto v(t) = t^{\nu} u(t)$ , where  $\nu = \frac{\alpha}{2} + \frac{1}{2}$  and  $u$  is a general solution to (2.8). Recall that the pair  $I_{\nu}$  and  $K_{\nu}$  of modified Bessel functions of the third kind always form a fundamental system of solutions to (2.8), see Watson [29, § 3.7]. Here

$$(2.9) \quad I_{\nu}(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m! \Gamma(\nu+m+1)}, \quad \nu \in \mathbb{C},$$

and

$$(2.10) \quad K_{\nu}(z) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \nu \pi}, \quad \nu \in \mathbb{C}.$$

Moreover, if  $\nu > 0$  then  $I_{\nu}(z)$  tends to infinity while  $K_{\nu}(z)$  tends exponentially to zero as  $z \rightarrow \infty$  through positive values, see Watson [29, § 7.23].

*Remark.* Equation (2.7) can be derived from Bessel's equation for functions of order  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ ,

$$(2.11) \quad z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2)u = 0,$$

by elementary transformations of the dependent and independent variables. Indeed, as we have seen we can transform (2.7) to (2.8), which differs from Bessel's equation only in the coefficient of  $u$ . By the change of variables  $t \mapsto it$ , (2.8) is transformed to Bessel's equation for functions of order  $\nu$ , so its general solution is given by  $z \mapsto u(iz)$  where  $u$  is a general solution to (2.11). This implies that the general solution to (2.7) can also be obtained as  $t \mapsto v(t) = t^{\nu} u(it)$ , where  $\nu = \frac{\alpha}{2} + \frac{1}{2}$  and  $u$  is a general solution to (2.11). Since the pair  $H_{\nu}^{(1)}$  and  $H_{\nu}^{(2)}$  of Bessel functions of the third kind always form a fundamental system of solutions to Bessel's equation for functions of order  $\nu$ , see Watson [29, § 3.63], it is thus possible to represent the Fourier transform of  $\mathcal{K}_{\alpha,y}$  also in terms of this pair of solutions (due to



considerations of growth, it turns out to be expressed by means of  $H_\nu^{(1)}$ ,  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ . We will not pursue this direction further.

**Theorem 2.6.** *Let  $\alpha > -1$ . Then the Fourier transform of  $\mathcal{K}_{\alpha,y}$  given by (2.2) can be expressed in terms of Bessel functions,*

$$(2.12) \quad \widehat{\mathcal{K}_{\alpha,y}}(\xi) = \frac{2^{(1-\alpha)/2}}{\Gamma((\alpha+1)/2)} (y|\xi|)^{(\alpha+1)/2} K_{(\alpha+1)/2}(y|\xi|), \quad y > 0,$$

where  $K_{(\alpha+1)/2}$  is the modified Bessel function of the third kind of order  $(\alpha+1)/2$ .

*Proof.* To shorten notation, let  $\nu = \frac{\alpha}{2} + \frac{1}{2}$  and note that  $\nu > 0$  by assumption. Let  $F_\alpha$  be the function satisfying  $F_\alpha(y|\xi|) = \widehat{\mathcal{K}_{\alpha,1}}(y\xi)$ . By the discussion preceding the theorem, it follows that since  $F_\alpha$  solves (2.7), we have

$$(2.13) \quad F_\alpha(r) = A_\alpha r^\nu I_\nu(r) + B_\alpha r^\nu K_\nu(r), \quad r \geq 0,$$

for some constants  $A_\alpha$  and  $B_\alpha$  which are to be determined. Since  $\widehat{\mathcal{K}_{\alpha,1}}$  is bounded by virtue of Lemma 2.3, and  $K_\nu(r)$  tends exponentially to zero while  $I_\nu(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , the coefficient of  $I_\nu$  in (2.13) must vanish, that is,  $A_\alpha = 0$ .

To determine  $B_\alpha$ , suppose first that  $\alpha > -1$  is not an odd positive integer, so that  $\nu \notin \mathbb{Z}_+$ . By an application of Theorem 2.4 we have that  $\widehat{\mathcal{K}_{\alpha,y}} \rightarrow 1$  in  $\mathcal{S}'$  as  $y \rightarrow 0$ . Since  $\widehat{\mathcal{K}_{\alpha,y}}(\xi)$  is a continuous function of  $\xi$  we find that  $\widehat{\mathcal{K}_{\alpha,y}}(\xi) \rightarrow 1$  as  $y \rightarrow 0$  for all  $\xi \in \mathbb{R}$ , which by virtue of (2.3) implies that  $\widehat{\mathcal{K}_{\alpha,1}}(y) = \widehat{\mathcal{K}_{\alpha,y}}(1) \rightarrow 1$  as  $y \rightarrow 0$ . In particular, this means that  $F_\alpha(r) \rightarrow 1$  as  $r \rightarrow 0$ . In view of (2.9) and (2.10) this gives

$$1 = \frac{B_\alpha 2^{\nu-1} \pi}{\sin \nu \pi \Gamma(1-\nu)}$$

since  $\nu > 0$ . Invoking Euler's reflection formula  $\Gamma(z)\Gamma(1-z)\sin \pi z = \pi$  we find that  $B_\alpha = 2^{1-\nu}/\Gamma(\nu)$ . Since  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ , this completes the proof in the case when  $\alpha$  is not an odd positive integer. In view of Definition 2.2, an application of the dominated convergence theorem shows that  $\widehat{\mathcal{K}_{\alpha,1}}(\xi) \rightarrow \widehat{\mathcal{K}_{2k-1,1}}(\xi)$  as  $\alpha \rightarrow 2k-1$ . By Watson [29, §3.7],  $K_\nu(z) \rightarrow K_k(z)$  as  $\nu = \frac{\alpha}{2} + \frac{1}{2} \rightarrow k$ , so the general case follows by continuity. This completes the proof.  $\square$

Note that since  $K_\nu(z)$  is an analytic function of  $z$ , Theorem 2.6 shows that the Fourier transform of  $\mathcal{K}_{\alpha,y}$  fails to be smooth at the origin. That  $\widehat{\mathcal{K}_{\alpha,y}}$  cannot be smooth on all of  $\mathbb{R}^n$  can of course also be seen directly from Definition 2.2 in view of how the Fourier transform relates integrability and regularity. We also remark that  $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ , so when  $\alpha \rightarrow 0$  we recover the Fourier transform of the Poisson kernel for the upper half space. Furthermore, a comparison with formulas (4.5) and (4.6) in Baldi et al. [5] (using  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ ) shows that we recover their result concerning the Fourier transform of the Poisson kernel of the infinitesimal generator associated to (1.2) in the case  $n = 1$ . The special case  $\alpha = n-1$  appears in Byczkowski et al. [9], see the proof of their Theorem 2.1.

Next, we give an integral representation of  $\widehat{\mathcal{K}_{\alpha,y}}$ . There is of course a wide variety of forms of the Bessel function  $K_\nu$  which can be used to express (2.12), but we will content ourselves with the following result which proves to be useful later.

**Corollary 2.7.** *Let  $\alpha > -1$ . Then the Fourier transform of  $\mathcal{K}_{\alpha,y}$  given by (2.2) can be expressed as*

$$\widehat{\mathcal{K}_{\alpha,y}}(\xi) = \frac{(y|\xi|)^{\alpha+1}}{\Gamma(\alpha+1)} \int_1^\infty e^{-y|\xi|t} (t^2 - 1)^{\alpha/2} dt, \quad y > 0.$$

*Proof.* By Watson [29, § 6.15], identity (4), we have the representation

$$K_\nu(z) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt.$$

In view of the duplication formula  $\Gamma(\nu)\Gamma(\nu + \frac{1}{2}) = 2^{1-2\nu}\Gamma(\frac{1}{2})\Gamma(2\nu)$ , the result is now an immediate consequence of Theorem 2.6.  $\square$

### 3. WEIGHTED SPACES OF DISTRIBUTIONS

In this section we recall certain facts concerning weighted spaces of distributions, and recall the definition of the  $\mathcal{S}'$ -convolution. We also prove some auxiliary results that will be used in the next section when we solve the Dirichlet problem (1.3). We mention that the weighted spaces of distributions that we will consider appear naturally in the context of Newtonian potentials of distributions, see Schwartz [25], and have subsequently been studied by many authors. They were recently used in Alvarez et al. [2] to characterize the tempered distributions that are  $\mathcal{S}'$ -convolvable with the classical Poisson kernel for the half space, and in Alvarez et al. [1] to study harmonic extensions of distributions. For further details on how these spaces appear, as well as on the  $\mathcal{S}'$ -convolution and other notions of convolution of tempered distributions, we refer to the mentioned papers and the references therein.

We begin by recalling the definitions and some properties of spaces of distributions considered by Laurent Schwartz. For details we refer to Schwartz [25, Chapitre VI, §8]. To make the notation less cumbersome we will as before assume that all the spaces under consideration below are defined on  $\mathbb{R}^n$  unless explicitly stated otherwise. We let  $\mathcal{D}_{L^p}$  denote the vector space of smooth functions  $\varphi \in \mathcal{C}^\infty$  such that all derivatives  $\partial^\beta \varphi$  belong to  $L^p$ . We endow  $\mathcal{D}_{L^p}$  with the topology in which a sequence  $\{\varphi_j\}_{j=1}^\infty$  converges to 0 in  $\mathcal{D}_{L^p}$  if  $\partial^\beta \varphi_j \rightarrow 0$  in  $L^p$  for all multi-indices  $\beta$ .  $\mathcal{D}_{L^p}$  is then a locally convex, complete topological vector space. We employ the notation  $\mathcal{B}$  for the special case  $p = \infty$ , that is,  $\mathcal{B} = \mathcal{D}_{L^\infty}$  is the space of smooth functions  $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $\varphi$  and all derivatives  $\partial^\beta \varphi$  are bounded. We will let  $\dot{\mathcal{B}}$  denote the closed subspace consisting of those elements  $\varphi \in \mathcal{B}$  such that  $\partial^\beta \varphi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  for all multi-indices  $\beta$ . We have the continuous strict inclusions

$$\mathcal{D}_{L^p} \subset \mathcal{D}_{L^q} \subset \dot{\mathcal{B}}, \quad 1 \leq p < q < \infty.$$

Moreover, the space  $\mathcal{C}_0^\infty$  of compactly supported smooth functions is dense in  $\mathcal{D}_{L^p}$  for  $1 \leq p < \infty$ , and in  $\dot{\mathcal{B}}$ , but  $\mathcal{C}_0^\infty$  is not dense in  $\mathcal{B}$ . For this reason we will also endow  $\mathcal{B}$  with the finest locally convex topology that on bounded subsets of  $\mathcal{B}$  induces the topology inherited from  $\mathcal{C}^\infty$ , and this space will be denoted by  $\mathcal{B}_c$ . We have that  $\mathcal{C}_0^\infty$  is dense in  $\mathcal{B}_c$ , and since  $\mathcal{C}_0^\infty$  is also dense in  $\dot{\mathcal{B}}$ , it follows that  $\dot{\mathcal{B}}$  is dense in  $\mathcal{B}_c$ .

For  $1 < p \leq \infty$  we let  $\mathcal{D}'_{L^p}$  denote the dual of  $\mathcal{D}_{L^{p'}}$  where  $p'$  is the conjugate exponent of  $p$ , and we let  $\mathcal{D}'_{L^1}$  denote the dual of  $\dot{\mathcal{B}}$ . Due to the dense inclusions  $\mathcal{C}_0^\infty \subset \mathcal{D}_{L^{p'}} \subset \dot{\mathcal{B}}$  for  $1 \leq p' < \infty$ , it follows that  $\mathcal{D}'_{L^p}$  is a space of distributions for

$1 \leq p \leq \infty$ . (This is not true for the dual of  $\mathcal{B}$ .) We have the continuous strict inclusions  $\mathcal{D}_{L^p} \subset L^p \subset \mathcal{D}'_{L^p}$ , as well as  $\mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q}$  for  $p < q$ .

By Schwartz [25, Chapitre VI, Théorème XXV], a distribution  $u$  belongs to  $\mathcal{D}'_{L^p}$  if and only if the regularization  $u * \varphi$  belongs to  $\mathcal{D}_{L^p}$  for every  $\varphi \in \mathcal{C}_0^\infty$ , where the convolution is defined in the usual distributional sense. Moreover,  $u$  belongs to  $\mathcal{D}'_{L^p}$  if and only if  $u$  can be represented as a finite sum

$$u = \sum_{\beta} \partial^{\beta} u_{\beta}, \quad u_{\beta} \in L^p,$$

where the derivatives are interpreted in the distributional sense. Hence,  $\mathcal{D}'_{L^p}$  is continuously embedded in the space  $\mathcal{S}'$  of tempered distributions. Consider now the case  $p = 1$ , and let  $u \in \mathcal{D}'_{L^1}$ . Representing  $u$  as a finite sum of distributional derivatives of integrable functions allows for  $u$  to be extended to a continuous linear form on  $\mathcal{B}_c$ . Since  $\mathcal{B}$  is dense in  $\mathcal{B}_c$ , the extension is unique. Hence  $\mathcal{D}'_{L^1}$  is the dual of  $\mathcal{B}_c$ , see Schwartz [25, p. 203]. Moreover, if  $\langle \cdot, \cdot \rangle_{\mathcal{V}', \mathcal{V}}$  in general denotes the duality pairing between a topological space  $\mathcal{V}$  and its dual  $\mathcal{V}'$ , the integral of  $u \in \mathcal{D}'_{L^1}$  is well defined in the sense that

$$\langle u, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} = \sum_{\beta} (-1)^{|\beta|} \int u_{\beta}(x) \partial^{\beta} 1 dx = \int u_0(x) dx.$$

If  $u \in \mathcal{D}'_{L^1}$  also belongs to  $L^1$ , then  $\langle u, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$  coincides with the integral of  $u$ . The distributions in  $\mathcal{D}'_{L^1}$  are therefore sometimes called integrable distributions.

**Definition 3.1.** Let  $w : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by  $w(x) = (1 + x^2)^{\frac{1}{2}}$ , and let  $\mu$  and  $p$  be real parameters with  $1 \leq p < \infty$ . We define the weighted space of distributions  $w^{\mu} \mathcal{D}'_{L^p}$  as

$$w^{\mu} \mathcal{D}'_{L^p} = \{u \in \mathcal{S}' : w^{-\mu} u \in \mathcal{D}'_{L^p}\}$$

with the topology induced by the map from  $w^{\mu} \mathcal{D}'_{L^p}$  to  $\mathcal{D}'_{L^p}$  given by  $u \mapsto w^{-\mu} u$ .

Note that  $(x, \xi) \mapsto w^{\mu}(\xi)$  belongs to the Kohn-Nirenberg symbol class of order  $\mu$ , that is, for any multi-indices  $\beta$  and  $\gamma$  we can find a constant  $C_{\beta, \gamma}$  such that

$$(3.1) \quad |\partial_x^{\beta} \partial_{\xi}^{\gamma} w^{\mu}(\xi)| \leq C_{\beta, \gamma} (1 + |\xi|)^{\mu - |\gamma|}, \quad \xi \in \mathbb{R}^n,$$

see Hörmander [18, Definition 18.1.1]. In particular,  $w^{\mu}$  is a so-called order function for any  $\mu \in \mathbb{R}$ , that is,  $\partial^{\beta} w^{\mu} = \mathcal{O}(w^{\mu})$  for any multi-index  $\beta$ . Since differentiation is a linear continuous operation on  $\mathcal{D}'_{L^p}$ , see Schwartz [25, p. 200], it follows that the space  $w^{\mu} \mathcal{D}'_{L^p}$  is also closed under differentiation.

For  $1 < p < \infty$ , the space  $w^{\mu} \mathcal{D}'_{L^p}$  is the dual of  $w^{-\mu} \mathcal{D}_{L^{p'}}$  where  $p'$  is the conjugate exponent of  $p$ . The space  $w^{\mu} \mathcal{D}'_{L^1}$  is the dual of  $w^{-\mu} \mathcal{B}$  and  $w^{-\mu} \mathcal{B}_c$ . Here the weighted spaces  $w^{-\mu} \mathcal{D}_{L^{p'}}$  are defined in the analog way as the functions  $\varphi \in \mathcal{C}^\infty$  such that  $w^{\mu} \varphi \in \mathcal{D}_{L^{p'}}$ . The duality is naturally given by

$$\langle u, \varphi \rangle_{w^{\mu} \mathcal{D}'_{L^p}, w^{-\mu} \mathcal{D}_{L^{p'}}} = \langle w^{-\mu} u, w^{\mu} \varphi \rangle_{\mathcal{D}'_{L^p}, \mathcal{D}_{L^{p'}}},$$

and similarly for  $w^{-\mu} \mathcal{B}$  and  $w^{-\mu} \mathcal{B}_c$ . However, since  $w^{\mu}$  is an order function it is easy to see that  $\varphi \in w^{\mu} \mathcal{D}_{L^p}$  if and only if  $w^{-\mu} \partial^{\beta} \varphi \in L^p$  for all multi-indices  $\beta$ , that is,

$$w^{\mu} \mathcal{D}_{L^p} = \mathcal{D}_{L^p(w^{-\mu p})},$$

where  $L^p(w^{-\mu p}) = L^p(w^{-\mu p}(x)dx)$  is the weighted  $L^p$  space defined in the natural way. It follows that  $L^p(w^{-\mu p})$  is continuously embedded in  $w^\mu \mathcal{D}'_{L^p}$  for  $\mu \in \mathbb{R}$  and  $1 \leq p < \infty$ . In particular

$$(3.2) \quad w^\mu \mathcal{D}'_{L^1} \subset L^1(w^{-\mu}) \subset w^\mu \mathcal{D}'_{L^1}, \quad \mu \in \mathbb{R}.$$

Conversely, we recall the following useful representation formula for distributions in  $w^\mu \mathcal{D}'_{L^p}$ , see Alvarez et al. [1, Lemma 3.3] and [1, Remark 3.4]. For  $\mu \in \mathbb{R}$  and  $1 \leq p < \infty$  we have that

$$(3.3) \quad w^\mu \mathcal{D}'_{L^p} = \left\{ u \in \mathcal{S}' : u = \sum_{\beta} \partial^\beta u_\beta, \quad u_\beta \in L^p(w^{-\mu p}) \right\},$$

where the summation is over a finite set. Pointwise multiplication is well defined and continuous from  $\mathcal{B} \times \mathcal{B}$  to  $\mathcal{B}$  and from  $\mathcal{B}' \times \mathcal{B}$  to  $\mathcal{B}'$ , so  $\mathcal{D}'_{L^1}$  is closed under multiplication by functions in  $\mathcal{B}$ . Using the representation formula (3.3) for  $p = 1$ , it follows that we have the continuous strict inclusions

$$w^{\mu_1} \mathcal{D}'_{L^1} \subset w^{\mu_2} \mathcal{D}'_{L^1} \subset \mathcal{S}', \quad \mu_1 < \mu_2.$$

We next recall the definition of the so-called  $\mathcal{S}'$ -convolution proposed by Hirata and Ogata [16]. The definition was given an equivalent form by Shiraishi [26], which is the one we will use.

**Definition 3.2.** Two tempered distributions  $u$  and  $v$  in  $\mathcal{S}'$  are said to be  $\mathcal{S}'$ -convolvable if the multiplicative product  $u(\check{v} * \varphi)$  belongs to  $\mathcal{D}'_{L^1}$  for every  $\varphi \in \mathcal{S}$ . Then the map from  $\mathcal{S}$  to  $\mathbb{C}$  given by

$$\varphi \mapsto \langle u(\check{v} * \varphi), 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$$

is linear and continuous, and thus defines a tempered distribution denoted by  $u * v$ .

Here,  $\check{\varphi}(x) = \varphi(-x)$  for  $\varphi \in \mathcal{S}$  and we extend this operation to  $\mathcal{S}'$  by duality. If  $\tau_\eta \varphi(x) = \varphi(x - \eta)$  denotes translation, then  $\check{v} * \varphi$  is the function

$$\eta \mapsto \langle \check{v}, \tau_\eta \check{\varphi} \rangle_{\mathcal{S}', \mathcal{S}} = \langle v, \tau_{-\eta} \varphi \rangle_{\mathcal{S}', \mathcal{S}}.$$

We remark that when defined, the  $\mathcal{S}'$ -convolution of  $u$  and  $v$  is commutative, and satisfies the Fourier exchange formula  $(u * v)^\wedge = \hat{u} \hat{v}$ . The notation  $u * v$  for the  $\mathcal{S}'$ -convolution of  $u$  and  $v$  is justified by the fact that Definition 3.2 coincides with the usual definition of convolution in the sense of distributions whenever the latter definition is applicable.

We now turn to the problem of finding the optimal class of tempered distributions that are  $\mathcal{S}'$ -convolvable with the kernel  $\mathcal{K}_{\alpha, y}$ . To simplify the notation below, we introduce the function  $w_\alpha$  given by

$$(3.4) \quad w_\alpha(x) = w^{\alpha+n+1}(x) = (1+x^2)^{(\alpha+n+1)/2}, \quad x \in \mathbb{R}^n.$$

We will thus not signify the dependence on the dimension in the notation. We remark that  $w_\alpha^{-1}$  is modulo a scaling factor equal to  $\mathcal{K}_{\alpha, 1}$ .

Our goal in this section is to prove the following result, which contains Alvarez et al. [2, Theorem 10] as the special case  $\alpha = 0$ .

**Theorem 3.3.** *Let  $\alpha > -1$ . Let  $w_\alpha$  be given by (3.4), and let  $f \in \mathcal{S}'$ . Then the following assertions are equivalent:*

- (i)  $f \in w_\alpha \mathcal{D}'_{L^1}$ .
- (ii)  $f$  is  $\mathcal{S}'$ -convolvable with  $\mathcal{K}_{\alpha, y}$  for each  $y > 0$ .

Furthermore, if  $f \in w_\alpha \mathcal{D}'_{L^1}$  then  $\mathcal{K}_{\alpha,y} * f$  is the function on  $\mathbb{R}^n$  given by

$$(3.5) \quad \eta \mapsto \langle w_\alpha^{-1} f, w_\alpha \tau_\eta \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$$

for each  $y > 0$ .

*Proof.* We shall adapt a combination of the proofs of Alvarez et al. [2, Proposition 7] and [2, Theorem 10] in the presence of a parameter  $\alpha > -1$ . Assume first that (i) holds, and let  $f = w_\alpha u$  for some  $u \in \mathcal{D}'_{L^1}$ . To prove that the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha,y} * f$  is well defined, we must according to Definition 3.2 show that the multiplicative product  $f(\mathcal{K}_{\alpha,y} * \varphi) = u(\mathcal{K}_{\alpha,y} * \varphi)w_\alpha$  belongs to  $\mathcal{D}'_{L^1}$  for each  $\varphi \in \mathcal{S}$  and  $y > 0$ . Since pointwise multiplication is well defined and continuous from  $\mathcal{B} \times \mathcal{B}$  to  $\mathcal{B}$  (and from  $\dot{\mathcal{B}} \times \dot{\mathcal{B}}$  to  $\dot{\mathcal{B}}$ ), it follows that  $\mathcal{D}'_{L^1}$  is closed under multiplication by functions in  $\mathcal{B}$ . It therefore suffices to show that we have  $(\mathcal{K}_{\alpha,y} * \varphi)w_\alpha \in \mathcal{B}$  for each  $\varphi \in \mathcal{S}$  and  $y > 0$ .

Recall Peetre's inequality

$$(3.6) \quad (1 + |t - x|^2)^s \leq 2^{|s|} (1 + x^2)^{|s|} (1 + t^2)^s, \quad s \in \mathbb{R}.$$

We have that  $\partial^\beta(\mathcal{K}_{\alpha,y} * \varphi) = \mathcal{K}_{\alpha,y} * \partial^\beta \varphi$ , and by using (3.6) it is straightforward to check that

$$|\mathcal{K}_{\alpha,y} * \partial^\beta \varphi(x)| \leq \frac{C_{\alpha,n}}{y^n} \left(1 + \frac{x^2}{y^2}\right)^{-(\alpha+n+1)/2} \int \left(1 + \frac{t^2}{y^2}\right)^{(\alpha+n+1)/2} \partial^\beta \varphi(t) dt,$$

where the constant  $C_{\alpha,n}$  depends on  $\alpha$  and  $n$ . Next, let  $M_{\alpha,n}$  denote the multiplication operator  $M_{\alpha,n}\varphi(t) = |t|^{\alpha+n+1}\varphi(t)$ . By splitting the integral in the right-hand side above into the two regions  $|t| < y$  and  $|t| \geq y$ , it is straightforward to check that this results in the estimate

$$|\mathcal{K}_{\alpha,y} * \partial^\beta \varphi(x)| \leq \frac{C_{\alpha,n}}{y^n} \left(1 + \frac{x^2}{y^2}\right)^{-(\alpha+n+1)/2} (\|\partial^\beta \varphi\|_{L^1} + y^{-(\alpha+n+1)} \|M_{\alpha,n} \partial^\beta \varphi\|_{L^1})$$

for some new constant  $C_{\alpha,n}$  depending on  $\alpha$  and  $n$ . Hence  $(\mathcal{K}_{\alpha,y} * \varphi)w_\alpha \in \mathcal{B}$ , so the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha,y} * f$  exists for each  $y > 0$  when  $f \in w_\alpha \mathcal{D}'_{L^1}$ . Moreover, we have

$$\langle \mathcal{K}_{\alpha,y} * f, \varphi \rangle_{\mathcal{S}', \mathcal{S}} = \langle f(\mathcal{K}_{\alpha,y} * \varphi), 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} = \langle w_\alpha^{-1} f, w_\alpha(\mathcal{K}_{\alpha,y} * \varphi) \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}.$$

Using (3.3) for  $w_\alpha^{-1} f = u \in \mathcal{D}'_{L^1}$ , it is straightforward to check that the quantity  $\langle u, w_\alpha(\mathcal{K}_{\alpha,y} * \varphi) \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$  coincides with the integral  $\int \langle u, w_\alpha \tau_\eta \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \varphi(\eta) d\eta$ , see the end of the proof of Alvarez et al. [2, Proposition 7] for details. This proves that the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha,y} * f$  is given by (3.5).

Assume next that (ii) holds, and fix  $f \in \mathcal{S}'$ . By Alvarez et al. [2, Proposition 9],  $f \in w_\alpha \mathcal{D}'_{L^1}$  if and only if  $f$  can be represented as a sum  $f = f_1 + M_{\alpha,n} f_2$ , where  $f_1 \in \mathcal{E}'$ ,  $M_{\alpha,n}$  is the multiplication operator introduced above, and  $f_2 \in \mathcal{D}'_{L^1}$  is not supported near the origin. Introduce a cutoff function  $\chi \in \mathcal{C}_0^\infty$  taking values in  $[0, 1]$  such that  $\chi(x)$  is identically equal to 1 for  $|x| \leq 1/2$ , positive for  $|x| < 1$  and vanishes for  $|x| \geq 1$ . Write  $f = \chi f + (1 - \chi)f$ , and note that  $f_1 = \chi f \in \mathcal{E}'$ .

Next, set  $\psi(t) = \chi(3t)$ . Then  $\psi(t) > 0$  for  $|t| < 1/3$ , and  $\psi(t) = 0$  for  $|t| \geq 1/3$ . In particular,  $\psi$  vanishes on  $\text{supp}(1 - \chi) = \{t \in \mathbb{R}^n : |t| \geq 1/2\}$ . Consider the convolution

$$\mathcal{K}_{\alpha,y} * \psi(x) = C_{\alpha,n} \int_{|t| < 1/3} \frac{y^{\alpha+1}}{(|x - t|^2 + y^2)^{(\alpha+n+1)/2}} \psi(t) dt.$$

For  $|t| < 1/3$  and  $x \in \text{supp}(1 - \chi)$  we have  $|x - t| \leq 2|x|$ , which implies that

$$\mathcal{K}_{\alpha,y} * \psi(x) \geq C_{\alpha,n} \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+n+1)/2}} \|\psi\|_{L^1}, \quad x \in \text{supp}(1 - \chi).$$

It follows that  $(1 - \chi)w_\alpha^{-1}(\mathcal{K}_{\alpha,y} * \psi)^{-1} \in \mathcal{B}$  for each  $y > 0$ . Moreover, assumption (ii) implies that we have  $(\mathcal{K}_{\alpha,y} * \psi)f \in \mathcal{D}'_{L^1}$  by virtue of Definition 3.2, which gives

$$\begin{aligned} (1 - \chi(x))f(x) &= |x|^{\alpha+n+1} \frac{w_\alpha(x)}{|x|^{\alpha+n+1}} \cdot \frac{1 - \chi(x)}{w_\alpha(x)\mathcal{K}_{\alpha,y} * \psi(x)} \mathcal{K}_{\alpha,y} * \psi(x)f(x) \\ &= M_{\alpha,n}(x)f_2(x), \end{aligned}$$

where  $f_2 \in \mathcal{D}'_{L^1}$  since the map

$$x \mapsto \frac{w_\alpha(x)}{|x|^{\alpha+n+1}} = (1 + |x|^{-2})^{(\alpha+n+1)/2}, \quad x \in \text{supp}(1 - \chi),$$

also belongs to  $\mathcal{B}$ , and  $\mathcal{D}'_{L^1}$  is closed under multiplication by functions in  $\mathcal{B}$ . Since  $f_2 \equiv 0$  near the origin, this completes the proof.  $\square$

*Remark.* We remark that the calculations in the proof of Theorem 3.3 show that if  $f \in w_\alpha \mathcal{D}'_{L^1}$  then the function  $F : \mathbb{R}_+^{n+1} \rightarrow \mathbb{C}$  given by

$$F(x, y) = \mathcal{K}_{\alpha,y} * f(x) = \langle w_\alpha^{-1}f, w_\alpha \tau_x \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

is in  $\mathcal{C}^\infty(\mathbb{R}_+^{n+1})$  and satisfies  $D_\alpha F = 0$  in  $\mathbb{R}_+^{n+1}$ . The derivatives  $\partial_x^\beta \partial_y^k F(x, y)$  are given by

$$\partial_x^\beta \partial_y^k F(x, y) = \langle w_\alpha^{-1}f, w_\alpha \partial_x^\beta \partial_y^k \tau_x \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}.$$

We end this section with the following proposition.

**Proposition 3.4.** *Let  $\alpha > -1$ . Let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . Then for any multi-index  $\beta$  we have  $\partial^\beta(\mathcal{K}_{\alpha,y} * f) = \mathcal{K}_{\alpha,y} * (\partial^\beta f)$  in  $\mathcal{S}'$  for each  $y > 0$ .*

*Proof.* Using the properties of the  $\mathcal{S}'$ -convolution we have

$$\begin{aligned} \langle \partial^\beta(\mathcal{K}_{\alpha,y} * f), \varphi \rangle_{\mathcal{S}', \mathcal{S}} &= (-1)^{|\beta|} \langle \mathcal{K}_{\alpha,y} * f, \partial^\beta \varphi \rangle_{\mathcal{S}', \mathcal{S}} \\ &= (-1)^{|\beta|} \langle f(\mathcal{K}_{\alpha,y} * \partial^\beta \varphi), 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \\ &= (-1)^{|\beta|} \langle f, \mathcal{K}_{\alpha,y} * \partial^\beta \varphi \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}, \end{aligned}$$

where the last expression makes sense, since  $f$  can be written  $f = w_\alpha u$  for some  $u \in \mathcal{D}'_{L^1}$ , and  $(\mathcal{K}_{\alpha,y} * \partial^\beta \varphi)w_\alpha$  belongs to  $\mathcal{B}$  by the first part of the proof of Theorem 3.3. Since  $\partial^\beta f \in w_\alpha \mathcal{D}'_{L^1}$ , we similarly have

$$\begin{aligned} \langle \mathcal{K}_{\alpha,y} * (\partial^\beta f), \varphi \rangle_{\mathcal{S}', \mathcal{S}} &= \langle \partial^\beta f(\mathcal{K}_{\alpha,y} * \varphi), 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \\ &= \langle \partial^\beta f, \mathcal{K}_{\alpha,y} * \varphi \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \\ &= (-1)^{|\beta|} \langle f, \mathcal{K}_{\alpha,y} * \partial^\beta \varphi \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}, \end{aligned}$$

which completes the proof.  $\square$

*Remark.* In Alvarez et al. [1, Lemma 2.4], a proof for the case  $\alpha = 0$  can be found utilizing cutoff functions and the formula (3.5).

## 4. THE DIRICHLET PROBLEM

In this section we show existence of solutions to the Dirichlet problem (1.3) for boundary data in the weighted space  $w_\alpha \mathcal{D}'_{L^1}$ .

**Definition 4.1.** Let  $\alpha > -1$ . Let  $w_\alpha$  be given by (3.4), and let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . The Poisson integral of  $f$  with respect to the kernel  $\mathcal{K}_\alpha$  is defined as the function

$$\mathcal{K}_\alpha[f] : (x, y) \mapsto \mathcal{K}_{\alpha, y} * f(x), \quad (x, y) \in \mathbb{R}_+^{n+1},$$

where the right-hand side is the  $\mathcal{S}'$ -convolution of  $\mathcal{K}_{\alpha, y}$  and  $f$ , and  $\mathcal{K}_{\alpha, y}(x) = \mathcal{K}_\alpha(x, y)$  in accordance with (1.4).

Henceforth, we will mostly write  $\mathcal{K}_\alpha[f]$  only when referring to the corresponding function on  $\mathbb{R}_+^{n+1}$ ; the value of  $\mathcal{K}_\alpha[f]$  at  $(x, y)$  will usually still be written as  $\mathcal{K}_{\alpha, y} * f(x)$ , and we will continue to write  $\mathcal{K}_{\alpha, y} * f$  when discussing the map  $x \mapsto \mathcal{K}_\alpha[f](x, y)$ . The next lemma describes the integrability properties of  $\mathcal{K}_{\alpha, y} * f$ .

**Lemma 4.2.** Let  $\alpha > -1$ . If  $f \in w_\alpha \mathcal{D}'_{L^1}$  then the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha, y} * f$  belongs to  $w_\alpha \mathcal{D}_{L^1}$  for all  $y > 0$ . In particular, the  $\mathcal{S}'$ -convolution with the kernel  $\mathcal{K}_{\alpha, y}$  preserves  $L^1(w_\alpha^{-1})$  for each  $y > 0$ .

*Proof.* As in the proof of Alvarez et al. [1, Lemma 3.1] we claim that it suffices to prove the implication

$$(4.1) \quad u \in w_\alpha \mathcal{D}'_{L^1} \implies \mathcal{K}_{\alpha, y} * u \in L^1(w_\alpha^{-1}).$$

Indeed, given  $f \in w_\alpha \mathcal{D}'_{L^1}$  we have that  $\partial^\beta(\mathcal{K}_{\alpha, y} * f) = \mathcal{K}_{\alpha, y} * (\partial^\beta f)$  for any multi-index  $\beta$  by Proposition 3.4. Since  $w_\alpha \mathcal{D}'_{L^1}$  is closed under differentiation we have  $\partial^\beta f \in w_\alpha \mathcal{D}'_{L^1}$ , so  $\mathcal{K}_{\alpha, y} * (\partial^\beta f)$  is smooth in view of the remark on page 14 following Theorem 3.3. Hence, if the implication (4.1) holds then

$$\partial^\beta(\mathcal{K}_{\alpha, y} * f) \in \mathcal{D}_{L^1(w_\alpha^{-1})} = w_\alpha \mathcal{D}_{L^1}$$

for any multi-index  $\beta$ , so  $\mathcal{K}_{\alpha, y} * f \in w_\alpha \mathcal{D}_{L^1}$ .

Thus, let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . By Definition 3.1 we can write  $f = \sum_\beta w_\alpha \partial^\beta f_\beta$  where  $f_\beta \in L^1$  and the sum is finite. We may therefore without loss of generality assume that  $f = w_\alpha \partial^\beta f_\beta$  with  $f_\beta \in L^1$ . According to (3.5) we then have

$$(4.2) \quad \begin{aligned} \|\mathcal{K}_{\alpha, y} * f\|_{L^1(w_\alpha^{-1})} &= \int w_\alpha^{-1}(\eta) |\langle \partial^\beta f_\beta, w_\alpha \tau_\eta \mathcal{K}_{\alpha, y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{D}_c}| d\eta \\ &\leq \int w_\alpha^{-1}(\eta) \int |f_\beta(x)| |\partial_x^\beta(w_\alpha(x) \mathcal{K}_{\alpha, y}(\eta - x))| dx d\eta. \end{aligned}$$

By Leibniz' formula

$$\partial_x^\beta(w_\alpha(x) \mathcal{K}_{\alpha, y}(\eta - x)) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma w_\alpha(x) \partial^{\beta-\gamma} \mathcal{K}_{\alpha, y}(\eta - x).$$

By using property (3.1) and the fact that  $1 + a \sim (1 + a^2)^{\frac{1}{2}}$  whenever  $a \geq 0$ , that is,  $(1 + a^2)^{\frac{1}{2}} \leq 1 + a \leq \sqrt{2}(1 + a^2)^{\frac{1}{2}}$  if  $a \geq 0$ , it is straightforward to check that

$$|\partial_x^\beta(w_\alpha(x) \mathcal{K}_{\alpha, y}(\eta - x))| \leq \sum_{\gamma \leq \beta} \frac{C_{\alpha, \beta, \gamma, n}}{y^{n+|\beta-\gamma|}} \cdot \frac{(1 + |x|)^{\alpha+n+1-|\gamma|}}{\left(1 + \frac{|\eta-x|}{y}\right)^{\alpha+n+1+|\beta-\gamma|}}.$$



Next, note that

$$\left(1 + \frac{|\eta - x|}{y}\right)^{-\alpha-n-1-|\beta-\gamma|} \leq (1 + |\eta - x|)^{-\alpha-n-1-|\beta-\gamma|} \max(1, y^{\alpha+n+1+|\beta-\gamma|}).$$

Hence, (4.2) and an application of Tonelli's theorem gives

$$\|\mathcal{K}_{\alpha,y} * f\|_{L^1(w_\alpha^{-1})} \leq \sum_{\gamma \leq \beta} C_{\alpha,\beta,\gamma,n,y} \int |f_\beta(x)| (1 + |x|)^{\alpha+n+1-|\gamma|} I(x) dx,$$

where

$$I(x) = \int w_\alpha^{-1}(\eta) (1 + |\eta - x|)^{-\alpha-n-1-|\beta-\gamma|} d\eta.$$

By Alvarez et al. [1, Lemma 2.8] we have  $0 < I(x) \leq C_{\alpha,\beta,\gamma,n} (1 + |x|)^{-\alpha-n-1}$ . Since  $f_\beta \in L^1$  we conclude that

$$\|\mathcal{K}_{\alpha,y} * f\|_{L^1(w_\alpha^{-1})} \leq \sum_{\gamma \leq \beta} C_{\alpha,\beta,\gamma,n,y} \int |f_\beta(x)| (1 + |x|)^{-|\gamma|} dx < \infty.$$

Having proved the first statement of the lemma, the last statement follows immediately by virtue of (3.2).  $\square$

By virtue of (3.2), Lemma 4.2 ensures that the  $\mathcal{S}'$ -convolution  $\mathcal{K}_{\alpha,y} * f$  belongs to  $w_\alpha \mathcal{D}'_{L^1}$  for all  $y > 0$  whenever  $f \in w_\alpha \mathcal{D}'_{L^1}$ . We can therefore consider the convergence of  $\mathcal{K}_{\alpha,y} * f$  in  $w_\alpha \mathcal{D}'_{L^1}$  as  $y \rightarrow 0$ .

**Theorem 4.3.** *Let  $\alpha > -1$ . Let  $f \in w_\alpha \mathcal{D}'_{L^1}$  and set  $u = \mathcal{K}_\alpha[f]$ . Then  $u_y \rightarrow f$  in  $w_\alpha \mathcal{D}'_{L^1}$  as  $y \rightarrow 0$ , where  $u_y(x) = u(x, y)$  for  $y > 0$  in accordance with (1.4).*

*Proof.* We will essentially adapt the proof of Alvarez et al. [1, Theorem 3.6]. Suppose first that we have already proved that if  $g \in L^1(w_\alpha^{-1})$  then  $\mathcal{K}_{\alpha,y} * g \rightarrow g$  in  $L^1(w_\alpha^{-1})$  as  $y \rightarrow 0$ . Since  $L^1(w_\alpha^{-1})$  is continuously embedded in  $w_\alpha \mathcal{D}'_{L^1}$ , it follows that  $\mathcal{K}_{\alpha,y} * g \rightarrow g$  in  $w_\alpha \mathcal{D}'_{L^1}$  as  $y \rightarrow 0$ . Now let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . By (3.3) we can then write  $f$  as a finite sum with terms of the form  $\partial^\beta f_\beta$  with  $f_\beta \in L^1(w_\alpha^{-1})$ . Recall that the operation of differentiation is continuous in  $w_\alpha \mathcal{D}'_{L^1}$  according to the discussion following Definition 3.1. By Proposition 3.4 we have  $\partial^\beta(\mathcal{K}_{\alpha,y} * f_\beta) = \mathcal{K}_{\alpha,y} * (\partial^\beta f_\beta)$  since  $f_\beta \in w_\alpha \mathcal{D}'_{L^1}$ . This gives

$$\mathcal{K}_{\alpha,y} * f = \sum_\beta \mathcal{K}_{\alpha,y} * (\partial^\beta f_\beta) = \sum_\beta \partial^\beta(\mathcal{K}_{\alpha,y} * f_\beta) \rightarrow \sum_\beta \partial^\beta f_\beta = f$$

in  $w_\alpha \mathcal{D}'_{L^1}$  as  $y \rightarrow 0$ . Hence it suffices to prove that if  $f \in L^1(w_\alpha^{-1})$  then  $\mathcal{K}_{\alpha,y} * f \rightarrow f$  in  $L^1(w_\alpha^{-1})$  as  $y \rightarrow 0$ .

Suppose therefore that  $f \in L^1(w_\alpha^{-1})$ . Since  $\alpha > -1$ , the definition of  $w_\alpha$  ensures that  $w_\alpha^{-1}(x)dx$  is a finite, complete, regular measure on  $\mathbb{R}^n$ , which implies that the compactly supported continuous functions  $\mathcal{C}_0$  are dense in  $L^1(w_\alpha^{-1})$ . Given  $\varepsilon > 0$  we let  $g \in \mathcal{C}_0$  satisfy

$$(4.3) \quad \|f - g\|_{L^1(w_\alpha^{-1})} < \varepsilon.$$

Next, note that since  $0 < w_\alpha^{-1}(x) \leq 1$  for  $x \in \mathbb{R}^n$ , it follows that

$$\begin{aligned} \|\mathcal{K}_{\alpha,y} * g - g\|_{L^1(w_\alpha^{-1})} &\leq \iint \mathcal{K}_{\alpha,y}(x) |g(t-x) - g(t)| dx dt \\ &= \int \mathcal{K}_{\alpha,y}(x) \|\tau_x g - g\|_{L^1} dx, \end{aligned}$$

where the last identity follows by Tonelli's theorem since  $\mathcal{K}_{\alpha,y}$  is nonnegative. It is well-known that since the unweighted  $L^1$  space is translation invariant, we have that  $\|\tau_x g - g\|_{L^1} \rightarrow 0$  as  $x \rightarrow 0$ , see Bochner [7, Theorem 1.2.1]. Since  $\mathcal{K}_{\alpha,y}$  enjoys the usual properties of kernel functions, that is,  $\int \mathcal{K}_{\alpha,y}(x)dx = 1$  for all  $y > 0$ , and for each  $r > 0$  we have

$$\lim_{y \rightarrow 0} \int_{|x| \geq r} \mathcal{K}_{\alpha,y}(x)dx = \lim_{y \rightarrow 0} \int_{y|x| \geq r} \mathcal{K}_{\alpha,1}(x)dx = 0,$$

it follows that

$$\|\mathcal{K}_{\alpha,y} * g - g\|_{L^1(w_\alpha^{-1})} \rightarrow 0 \quad \text{as } y \rightarrow 0,$$

see Bochner [7, Theorem 1.3.2]. Now,

$$(4.4) \quad \begin{aligned} \|\mathcal{K}_{\alpha,y} * f - f\|_{L^1(w_\alpha^{-1})} &\leq \|\mathcal{K}_{\alpha,y} * (f - g)\|_{L^1(w_\alpha^{-1})} \\ &\quad + \|\mathcal{K}_{\alpha,y} * g - g\|_{L^1(w_\alpha^{-1})} + \|g - f\|_{L^1(w_\alpha^{-1})}, \end{aligned}$$

so it only remains to estimate the first term in the right-hand side. Note that

$$(4.5) \quad \|\mathcal{K}_{\alpha,y} * (f - g)\|_{L^1(w_\alpha^{-1})} \leq \int |f(t) - g(t)| \mathcal{K}_{\alpha,y} * w_\alpha^{-1}(t) dt$$

by Tonelli's theorem and the fact that both  $\mathcal{K}_{\alpha,y}$  and  $w_\alpha^{-1}$  are radial functions.

We now estimate  $\mathcal{K}_{\alpha,y} * w_\alpha^{-1}(t)$ . It is straightforward to check that the function  $y \mapsto \mathcal{K}_{\alpha,y}(x)$  is increasing for  $0 < y < |x|/\varrho$  where  $\varrho = \varrho_{\alpha,n} = \sqrt{n}/\sqrt{\alpha+1}$ . Hence, if  $y < 1$  and  $|x| \geq 2\varrho$ , then  $\mathcal{K}_{\alpha,y}(x) \leq \mathcal{K}_{\alpha,y+1}(x)$ . This gives

$$\begin{aligned} \mathcal{K}_{\alpha,y} * w_\alpha^{-1}(t) &\leq \int_{|x| \geq 2\varrho} \mathcal{K}_{\alpha,y+1}(x) w_\alpha^{-1}(t-x) dx \\ &\quad + \int_{|x| < 2\varrho} \mathcal{K}_{\alpha,y}(x) w_\alpha^{-1}(t-x) dx = I_1(t) + I_2(t). \end{aligned}$$

Below we use the fact that  $1 + a \sim (1 + a^2)^{\frac{1}{2}}$  whenever  $a \geq 0$ , and we let  $C_{\alpha,n}$  denote a constant, depending on  $\alpha$  and  $n$ , the value of which is permitted to change between occurrences. We first estimate  $I_1(t)$ . For  $0 < y < 1$  we have  $\mathcal{K}_{\alpha,y+1}(x) \leq 2^{\alpha+1} \mathcal{K}_{\alpha,1}(x)$ , which gives

$$I_1(t) \leq C_{\alpha,n} \int_{|x| \geq 2\varrho} (1 + |x|)^{-(\alpha+n+1)/2} (1 + |t-x|)^{-(\alpha+n+1)/2} dx.$$

Since  $\alpha > -1$ , it follows by an application of Alvarez et al. [1, Lemma 2.8] that the right-hand side is finite and bounded by a constant  $C_{\alpha,n}$  multiplied by  $w_\alpha^{-1}(t)$ . To estimate  $I_2(t)$  we apply Peetre's inequality (3.6) to  $w_\alpha^{-1}(t-x)$  which gives

$$\begin{aligned} I_2(t) &\leq C_{\alpha,n} w_\alpha^{-1}(t) \int_{|x| < 2\varrho} \mathcal{K}_{\alpha,y}(x) w_\alpha(x) dx \\ &\leq C_{\alpha,n} w_\alpha(2\varrho) w_\alpha^{-1}(t) \int \mathcal{K}_{\alpha,y}(x) dx = C_{\alpha,n} w_\alpha^{-1}(t). \end{aligned}$$

Combining the estimates for  $I_1(t)$  and  $I_2(t)$  we thus have  $0 < \mathcal{K}_{\alpha,y} * w_\alpha^{-1}(t) \leq C_{\alpha,n} w_\alpha^{-1}(t)$ . By virtue of (4.5) we find that  $\|\mathcal{K}_{\alpha,y} * (f - g)\|_{L^1(w_\alpha^{-1})} \leq C_{\alpha,n} \|f - g\|_{L^1(w_\alpha^{-1})}$ . In view of (4.3)–(4.4) this implies that  $\|\mathcal{K}_{\alpha,y} * f - f\|_{L^1(w_\alpha^{-1})} \leq (C_{\alpha,n} + 2)\varepsilon$  for any sufficiently small  $y > 0$ , which completes the proof.  $\square$

**Corollary 4.4.** *Let  $\alpha > -1$ . Let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . Then  $\mathcal{K}_\alpha[f]$  is a solution to the Dirichlet problem (1.3).*

*Proof.* Set  $u = \mathcal{K}_\alpha[f]$  and note that  $w_\alpha \mathcal{D}'_{L^1} \subset \mathcal{S}'$  with continuous inclusion. By Theorem 4.3 we thus have  $u_y \rightarrow f$  in  $\mathcal{S}'$  as  $y \rightarrow 0$ , where  $u_y(x) = u(x, y)$  for  $y > 0$  in accordance with (1.4). In view of the remark on page 14, we have  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$ , which completes the proof.  $\square$

Compared to the harmonic case  $\alpha = 0$ , the proof of Theorem 4.3 is complicated by the fact that when  $\alpha \neq 0$ , the family  $\{\mathcal{K}_{\alpha,y}\}_{y>0}$  does not in general satisfy the semi-group property  $\mathcal{K}_{0,y_1+y_2} = \mathcal{K}_{0,y_1} * \mathcal{K}_{0,y_2}$  enjoyed by the classical Poisson kernel  $\mathcal{K}_0$ . (This would have allowed for an easier estimation of  $\mathcal{K}_{\alpha,y} * w_\alpha^{-1}(t)$ .) In fact, in the context of hyperbolic Brownian motion it is a point of interest to determine, for each fixed  $\eta > 0$ , the function  $\mathcal{G}_\alpha \equiv \mathcal{G}_\alpha(\eta)$  such that  $\mathcal{G}_\alpha : (x, y) \mapsto \mathcal{G}_\alpha(x, y)$  satisfies

$$(4.6) \quad \mathcal{K}_{\alpha,y} = \mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}, \quad y > \eta,$$

where  $\mathcal{G}_{\alpha,y}(x) = \mathcal{G}_\alpha(x, y)$  in accordance with (1.4).  $\mathcal{G}_\alpha$  is then the kernel function for the Dirichlet problem  $D_\alpha u = 0$  in the half space  $y > \eta$  with boundary conditions given on the hyperplane  $y = \eta$ . In other words, it is the probability density function of the measure of probability that the process (1.2) with  $\mu = \frac{\alpha}{2} + \frac{1}{2}$  and starting at  $(0, y)$ ,  $y > \eta$ , hits a portion of the boundary  $y = \eta$ . (For this probability measure, the terminology *hitting distribution* is commonly used.) We would thus like to solve (4.6), a priori interpreted in the distributional sense by means of the  $\mathcal{S}'$ -convolution, and find a solution in  $w_\alpha \mathcal{D}'_{L^1}$  to justify the equation. Since the  $\mathcal{S}'$ -convolution satisfies the Fourier exchange formula, a necessary condition is that  $\widehat{\mathcal{K}_{\alpha,y}} = \widehat{\mathcal{G}_{\alpha,y}} \widehat{\mathcal{K}_{\alpha,\eta}}$ . Since the Fourier transform of  $\mathcal{K}_{\alpha,y}$  is nonvanishing for each  $y > 0$ , this is equivalent to

$$(4.7) \quad \widehat{\mathcal{G}_{\alpha,y}}(\xi) = \frac{\widehat{\mathcal{K}_{\alpha,y}}(\xi)}{\widehat{\mathcal{K}_{\alpha,\eta}}(\xi)} = \left(\frac{y}{\eta}\right)^{(\alpha+1)/2} \frac{K_{(\alpha+1)/2}(y|\xi|)}{K_{(\alpha+1)/2}(\eta|\xi|)}, \quad y > \eta,$$

where the second formula follows from Theorem 2.6. As before,  $K_{(\alpha+1)/2}$  denotes the modified Bessel function of the third kind of order  $\frac{\alpha}{2} + \frac{1}{2}$ . For  $n = 1$ , this formula appears for example in Baldi et al. [5, Section 4] (choosing  $y = 1$  and  $\nu = \frac{\alpha}{2} + \frac{1}{2}$ ). An equivalent formula also appears in Byczkowski et al. [9, Theorem 2.1] in the special case  $\alpha = n - 1$ . In both instances the proofs involve probabilistic methods.

Using Corollary 2.7 together with the first identity in (4.7) we have

$$\widehat{\mathcal{G}_{\alpha,y}}(\xi) = \left(\frac{y}{\eta}\right)^{\alpha+1} \frac{\int_1^\infty e^{-y|\xi|t} (t^2 - 1)^{\alpha/2} dt}{\int_1^\infty e^{-\eta|\xi|t} (t^2 - 1)^{\alpha/2} dt} \leq \left(\frac{y}{\eta}\right)^{\alpha+1} e^{-(y-\eta)|\xi|}, \quad y > \eta,$$

which shows that  $\widehat{\mathcal{G}_{\alpha,y}}$  is rapidly decreasing and belongs to  $L^2$ . By means of the Fourier inversion formula, this gives an element  $\mathcal{G}_{\alpha,y} \in \mathcal{C}^\infty$ , uniquely defined in  $L^2$ . Since  $\mathcal{K}_{\alpha,\eta} \in L^1$  by Lemma 2.3, the convolution  $\mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}$  is well defined in the usual sense; moreover, it belongs to  $L^2$  by Young's inequality and the Fourier exchange formula holds, so

$$(4.8) \quad \widehat{\mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}} = \widehat{\mathcal{G}_{\alpha,y}} \widehat{\mathcal{K}_{\alpha,\eta}} = \widehat{\mathcal{K}_{\alpha,y}}.$$

It follows that  $\mathcal{G}_\alpha$  is a solution to (4.6). Indeed, it is straightforward to check that the translation  $\tau_h \mathcal{K}_{\alpha,\eta}$  tends to  $\mathcal{K}_{\alpha,\eta}$  in  $L^2$  as  $h \rightarrow 0$ , and that  $\mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}$  therefore

is continuous. By Parseval's formula and (4.8) we have

$$\int \mathcal{K}_{\alpha,y}(x)\varphi(x)dx = \int \widehat{\mathcal{K}_{\alpha,y}}(\xi)\hat{\varphi}(-\xi)d\xi = \int \mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}(x)\varphi(x)dx$$

for all  $\varphi \in \mathcal{C}_0^\infty$ , which implies that  $\mathcal{K}_{\alpha,y} = \mathcal{G}_{\alpha,y} * \mathcal{K}_{\alpha,\eta}$ , see Hörmander [17, Theorem 1.2.4].

Note that the Fourier transform of  $\mathcal{G}_{\alpha,y}$  is radial by (4.7). In view of the discussion surrounding (2.4), the Fourier inversion formula can therefore be used to obtain the representation formula

$$\mathcal{G}_{\alpha,y}(x) = \frac{|x|^{(2-n)/2}}{(2\pi)^{n/2}} \left(\frac{y}{\eta}\right)^{(\alpha+1)/2} \int_0^\infty \frac{K_{(\alpha+1)/2}(ys)}{K_{(\alpha+1)/2}(\eta s)} s^{n/2} J_{(n-2)/2}(|x|s) ds.$$

This formula appears in Byczkowski et al. [9, Theorem 2.2] in the special case  $\alpha = n - 1$ . Writing  $\mathcal{G}_{\alpha,y}(\eta)$  for the function appearing above, they also show that when  $\alpha = n - 1$ , the family  $\{\mathcal{G}_{\alpha,y}(\eta)\}_{0 < \eta < y}$  satisfies the semi-group property

$$\mathcal{G}_{\alpha,y}(\eta_1) = \mathcal{G}_{\alpha,y}(\eta_2) * \mathcal{G}_{\alpha,\eta_2}(\eta_1), \quad 0 < \eta_1 < \eta_2 < y.$$

Clearly, this continues to hold for arbitrary  $\alpha > -1$ ; in fact, it is an immediate consequence of the first identity in (4.7) in view of the previous discussion. For completeness we collect these observations in the following proposition.

**Proposition 4.5.** *Let  $\alpha > -1$  and let  $\mathcal{K}_\alpha$  be given by Definition 2.2. For each  $\eta > 0$  there is a uniquely defined function  $\mathcal{G}_\alpha(\eta)$  satisfying  $\mathcal{K}_{\alpha,y} = \mathcal{G}_{\alpha,y}(\eta) * \mathcal{K}_{\alpha,\eta}$  for  $y > \eta$ .  $\mathcal{G}_\alpha(\eta)$  can be represented by*

$$\mathcal{G}_\alpha(\eta) : (x, y) \mapsto \frac{|x|^{(2-n)/2}}{(2\pi)^{n/2}} \left(\frac{y}{\eta}\right)^{(\alpha+1)/2} \int_0^\infty \frac{K_{(\alpha+1)/2}(ys)}{K_{(\alpha+1)/2}(\eta s)} s^{n/2} J_{(n-2)/2}(|x|s) ds,$$

where  $J_\nu$  denotes the Bessel function of the first kind of order  $\nu$ . Moreover, the Fourier transform of  $\mathcal{G}_{\alpha,y}(\eta)$  is given by (4.7), and the family  $\{\mathcal{G}_{\alpha,y}(\eta)\}_{0 < \eta < y}$  satisfies the semi-group property

$$\mathcal{G}_{\alpha,y}(\eta_1) = \mathcal{G}_{\alpha,y}(\eta_2) * \mathcal{G}_{\alpha,\eta_2}(\eta_1), \quad 0 < \eta_1 < \eta_2 < y.$$

## 5. ASYMPTOTIC BEHAVIOR OF THE POISSON INTEGRAL

In this section we investigate asymptotic growth behavior of the Poisson integral  $\mathcal{K}_\alpha[f]$  when  $f \in w_\alpha \mathcal{D}'_{L^1}$ . We shall obtain growth estimates comparable to those satisfied by the classical ( $\alpha = 0$ ) Poisson integral of  $p$ -summable functions proved by Siegel and Talvila [27]. We begin with a lemma, which is essentially just a version of [27, Theorem 2.1] in the presence of additional parameters, proved using similar techniques. We shall write  $S_r$  to denote the set

$$S_r = \mathbb{R}_+^{n+1} \cap \{(x, y) \in \mathbb{R}^{n+1} : x^2 + y^2 = r^2\}, \quad r > 0.$$

**Lemma 5.1.** *Let  $\alpha > -1$ . For each  $k \in \mathbb{N}$ , define  $L_{\alpha,k} : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$  by*

$$L_{\alpha,k}(x, y) = \frac{y^{\alpha+1}}{(x^2 + y^2)^{(\alpha+n+1+k)/2}}.$$

(Thus, modulo a scaling factor we have  $\mathcal{K}_\alpha = L_{\alpha,0}$ .) If  $\mu \leq \alpha + n + 1$  and  $f \in L^1(w^{-\mu})$  then

$$\int L_{\alpha,k}(x - \eta, y) |f(\eta)| d\eta \leq C_\mu I(r) r^{\mu-n-k} \sec^{n+k} \vartheta, \quad (x, y) \in S_r,$$

where  $I(r) \rightarrow 0$  as  $r \rightarrow \infty$ , and  $\vartheta \in [0, \pi/2)$  is the angle defined by  $y = r \cos \vartheta$  and  $|x| = r \sin \vartheta$  for  $(x, y) \in S_r$ .

*Proof.* Introduce the angle  $\omega$  defined by  $\langle x, \eta \rangle = |x||\eta| \cos \omega$  for  $x$  and  $\eta$  in  $\mathbb{R}^n$ . By the definition of the angle  $\vartheta$  we then have  $\langle x, \eta \rangle = |\eta| r \sin \vartheta \cos \omega$ , which gives

$$\begin{aligned} |x - \eta|^2 + y^2 &= (|\eta| - r)^2 + 2|\eta|r - 2\langle x, \eta \rangle \\ &= (1 - \sin \vartheta \cos \omega) \left( \frac{(|\eta| - r)^2}{1 - \sin \vartheta \cos \omega} + 2|\eta|r \right). \end{aligned}$$

It is straightforward to check that

$$\frac{\cos^2 \vartheta}{1 + \sin \vartheta} \leq 1 - \sin \vartheta \cos \omega \leq 2,$$

and using these bounds we obtain the estimate

$$|x - \eta|^2 + y^2 \geq \frac{\cos^2 \vartheta}{1 + \sin \vartheta} \left( \frac{(|\eta| - r)^2}{2} + 2|\eta|r \right) = \frac{\cos^2 \vartheta}{1 + \sin \vartheta} \cdot \frac{(|\eta| + r)^2}{2}.$$

Hence, for  $\mu \geq 0$  we have

$$(5.1) \quad (|x - \eta|^2 + y^2)^{-\mu/2} \leq \frac{(1 + \sin \vartheta)^{\mu/2}}{\cos^\mu \vartheta} \cdot \frac{2^{\mu/2}}{(|\eta| + r)^\mu} \leq \frac{2^\mu}{\cos^\mu \vartheta} (\eta^2 + r^2)^{-\mu/2},$$

where we in the last inequality also use the fact that  $(a^2 + b^2)^{\frac{1}{2}} \leq a + b$  when  $a$  and  $b$  are positive real numbers. Now write

$$\int L_{\alpha,k}(x - \eta, y) |f(\eta)| d\eta = \int \frac{|f(\eta)|}{(|x - \eta|^2 + y^2)^{\mu/2}} \cdot \frac{y^{\alpha+1} d\eta}{(|x - \eta|^2 + y^2)^{(\alpha+n+1+k-\mu)/2}}$$

and note that

$$y^{\alpha+1} \sup_{\eta} (|x - \eta|^2 + y^2)^{-(\alpha+n+1+k-\mu)/2} = y^{\mu-n-k}$$

for  $k \in \mathbb{N}$  when  $\mu \leq \alpha + n + 1$ . Hence, by virtue of (5.1) we find that

$$\int L_{\alpha,k}(x - \eta, y) |f(\eta)| d\eta \leq 2^\mu I(r) y^{\mu-n-k} \sec^\mu \vartheta,$$

where

$$I(r) = \int \frac{|f(\eta)|}{(\eta^2 + r^2)^{\mu/2}} d\eta \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

by the dominated convergence theorem since  $I(r) \leq \|f\|_{L^1(w^{-\mu})}$  when  $r = (x^2 + y^2)^{\frac{1}{2}} \geq 1$ . Recalling that for  $(x, y) \in S_r$  we have  $y = r \cos \vartheta$  we obtain

$$\int L_{\alpha,k}(x - \eta, y) |f(\eta)| d\eta \leq 2^\mu I(r) r^{\mu-n-k} \sec^{n+k} \vartheta,$$

which yields the result.  $\square$

Before using Lemma 5.1 to obtain growth estimates of the Poisson integral  $\mathcal{K}_\alpha[f]$  for general  $f \in w_\alpha \mathcal{D}'_{L^1}$ , we mention that an application of the lemma with  $k = 0$  and  $\mu = \alpha + n + 1$  shows that if  $f \in L^1(w_\alpha^{-1})$  then

$$(5.2) \quad \sup_{S_r} |\mathcal{K}_{\alpha,y} * f(x) \cos^n \vartheta| = o(r^{\alpha+1}) \quad \text{as } r \rightarrow \infty.$$

When  $\alpha = 0$  we recover the corresponding result of Siegel and Talvila [27, Corollary 2.1] for the usual Poisson integral of elements in  $L^1(w^{-(n+1)})$ . In analogy, the order relation (5.2) is sharp in the sense that the exponents cannot in general be decreased. To prove this, the arguments used for  $\alpha = 0$  by Siegel and Talvila [27, p. 576] are adapted to handle the full parameter range  $\alpha > -1$ .

Let  $\hat{e}_1$  be the unit vector along the  $x_1$  axis. Let  $f_k$ ,  $a_k$  and  $\varrho_k$  be positive real numbers such that  $\varrho_k < 1$ ,  $a_k \rightarrow \infty$  as  $k \rightarrow \infty$  and the balls  $B_{\varrho_k}(a_k \hat{e}_1) \subset \mathbb{R}^n$  with center at  $a_k \hat{e}_1$  and radius  $\varrho_k$  are disjoint. Define a continuous function  $f$  vanishing outside these balls by

$$f(x) = \begin{cases} f_k(1 - \frac{1}{\varrho_k}|x - a_k \hat{e}_1|), & x \in B_{\varrho_k}(a_k \hat{e}_1), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that  $f \in L^1(w_\alpha^{-1})$  if and only if

$$(5.3) \quad \sum_k f_k \frac{\varrho_k^n}{a_k^{\alpha+n+1}} < \infty,$$

and that if  $u(x, y) = \mathcal{K}_\alpha[f](x, y)$ , then  $u$  can be written as a superposition

$$u(x, y) = \sum_k f_k \int_{B_1(0)} (1 - |\eta|) \mathcal{K}_{\alpha, y} \left( \frac{x - a_k \hat{e}_1}{\varrho_k} - \eta, \frac{y}{\varrho_k} \right) d\eta$$

of translates of the function  $\tilde{u}(x, y) = \mathcal{K}_{\alpha, y} * \max(0, 1 - |\cdot|)(x)$ . Here  $B_1(0)$  is the unit ball in  $\mathbb{R}^n$ . Note that  $D_\alpha \tilde{u} = 0$  in  $\mathbb{R}_+^{n+1}$  and  $\tilde{u}(x, 0) = \max(0, 1 - |x|)$ . Since  $\tilde{u} \geq 0$  we thus have

$$u(x, y) \geq f_k \tilde{u} \left( \frac{x - a_k \hat{e}_1}{\varrho_k}, \frac{y}{\varrho_k} \right), \quad k \geq 1.$$

We now claim that if  $\beta + \gamma < \alpha + n + 1$ , or  $\beta + \gamma = \alpha + n + 1$  but  $\gamma < n$ , then  $r^{-\beta} u(x, y) \cos^\gamma \vartheta$  does not tend to zero along the sequence  $(x^{(k)}, y^{(k)}) = (a_k \hat{e}_1, \varrho_k)$  for appropriate choices of the numbers  $f_k$ ,  $a_k$  and  $\varrho_k$ . Indeed, if  $\beta + \gamma < \alpha + n + 1$ , set  $a_k = e^k$ ,  $f_k = e^{k(\alpha+n+1)}$  and  $\varrho_k = k^{-2}$ . Then the series (5.3) is easily seen to be convergent, while

$$\frac{(y^{(k)})^\gamma u(x^{(k)}, y^{(k)})}{((x^{(k)})^2 + (y^{(k)})^2)^{(\beta+\gamma)/2}} \geq \frac{\varrho_k^\gamma f_k \tilde{u}(0, 1)}{(a_k^2 + \varrho_k^2)^{(\beta+\gamma)/2}} = \frac{k^{-2\gamma} e^{k(\alpha+n+1)} \tilde{u}(0, 1)}{(e^{2k} + k^{-4})^{(\beta+\gamma)/2}}$$

does not tend to zero as  $k \rightarrow \infty$  since  $\beta + \gamma < \alpha + n + 1$  and

$$\tilde{u}(0, 1) = \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2) \pi^{n/2}} \int_{B_1(0)} \frac{(1 - |\eta|)}{(1 + \eta^2)} d\eta \neq 0.$$

If instead  $\beta + \gamma = \alpha + n + 1$ , but  $\gamma < n$ , let  $\varepsilon = n - \gamma > 0$  and set  $a_k = e^k$ ,  $f_k = e^{k(\alpha+n+1)} k^{\gamma(1+\varepsilon)/\varepsilon}$  and  $\varrho_k = k^{-(1+\varepsilon)/\varepsilon}$ . Then the left-hand side of (5.3) is equal to  $\sum_k k^{-1-\varepsilon} < \infty$ . However,  $\varrho_k^\gamma f_k = e^{k(\alpha+n+1)}$ , so

$$\frac{(y^{(k)})^\gamma u(x^{(k)}, y^{(k)})}{((x^{(k)})^2 + (y^{(k)})^2)^{(\beta+\gamma)/2}} \geq \frac{e^{k(\alpha+n+1)} \tilde{u}(0, 1)}{(e^{2k} + k^{-2(1+\varepsilon)/\varepsilon})^{(\alpha+n+1)/2}}$$

does not tend to zero as  $k \rightarrow \infty$ . Thus, the order relation (5.2) is sharp.

**Theorem 5.2.** *Let  $\alpha > -1$ . Let  $f \in w_\alpha \mathcal{D}'_{L^1}$ . Then there exists a nonnegative integer  $m$  depending only on the distribution  $f$  such that the Poisson integral  $\mathcal{K}_\alpha[f]$  satisfies*

$$\sup_{S_r} |\mathcal{K}_{\alpha,y} * f(x) \cos^{n+m} \vartheta| = o(r^{\alpha+1}) \quad \text{as } r \rightarrow \infty,$$

where  $\vartheta \in [0, \pi/2)$  is the angle defined by  $y = r \cos \vartheta$  and  $|x| = r \sin \vartheta$  for  $(x, y) \in S_r$ .

*Proof.* By the representation formula (3.3) we have  $f = \sum_{|\beta| \leq m} \partial^\beta f_\beta$  where  $f_\beta \in L^1(w_\alpha^{-1})$  and  $m \in \mathbb{N}$ . Note that

$$|\partial_\eta^\beta \mathcal{K}_{\alpha,y}(x - \eta)| \leq C_{\beta,\alpha,n} \frac{y^{\alpha+1}}{(|x - \eta|^2 + y^2)^{(\alpha+n+1+|\beta|)/2}} = C_{\beta,\alpha,n} L_{\alpha,|\beta|}(x - \eta, y),$$

where  $L_{\alpha,k}$  is the function defined in Lemma 5.1 for  $k \in \mathbb{N}$ . Thus

$$\begin{aligned} \mathcal{K}_{\alpha,y} * f(x) &= \sum_{|\beta| \leq m} \langle w_\alpha^{-1} \partial^\beta f_\beta, w_\alpha \tau_x \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \\ &= \sum_{|\beta| \leq m} (-1)^{|\beta|} \langle f_\beta, \partial^\beta \tau_x \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \end{aligned}$$

where each term  $\langle f_\beta, \partial^\beta \tau_x \mathcal{K}_{\alpha,y} \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$  in the right-hand side can be identified with the corresponding integral  $\int f_\beta(\eta) \partial_\eta^\beta \mathcal{K}_{\alpha,y}(x - \eta) d\eta$  in view of Lemma 5.1. Applying the lemma with  $\mu = \alpha + n + 1$  gives the estimate

$$\begin{aligned} |\mathcal{K}_{\alpha,y} * f(x)| &\leq \sum_{|\beta| \leq m} C_\beta r^{\alpha+1-|\beta|} \sec^{n+|\beta|} \vartheta I_\beta(r) \\ &\leq C_m r^{\alpha+1} \sec^{n+m} \vartheta \sum_{k=0}^m r^{-k} \sec^{k-m} \vartheta R_k(r), \end{aligned}$$

where  $R_k(r) = \sum_{|\beta|=k} I_\beta(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Since  $|r^{-k} \sec^{k-m} \vartheta| \leq 1$  for  $0 \leq k \leq m$  and  $r \geq 1$ , this completes the proof.  $\square$

As a final note, we shall briefly discuss the question of uniqueness of solutions to (1.3). The result by Huber [20] included above as Theorem 2.1 makes evident that a growth condition at infinity is needed in order to have uniqueness of solutions for the Dirichlet problem (1.3). In fact, the function

$$u(x, y) = y^{\alpha+1}, \quad (x, y) \in \mathbb{R}_+^{n+1},$$

satisfies  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$  and vanishes on the boundary. In analogy with the unweighted case, solutions to (1.3) satisfy the following principle of Phragmén-Lindelöf type, also due to Huber [19]. The result is stated verbatim but using our choice of notation.

**Theorem 5.3** (A. Huber). *Let  $u$  be a solution of  $D_\alpha u = 0$  ( $\alpha > -1$ ), defined in  $\mathbb{R}_+^{n+1}$  and satisfying at the boundary*

$$\limsup_{(x,y) \rightarrow (x_0,0)} u(x, y) \leq 0 \quad ((x, y) \in \mathbb{R}_+^{n+1}; x_0 \in \mathbb{R}^n).$$

*If follows that*

- (a) *the limit  $\varrho = \lim_{r \rightarrow \infty} m(r)/r^{\alpha+1}$ , where  $m(r) = \sup_{(x,y) \in S_r} u(x, y)$ , always exists (finite or infinite),*



- (b)  $\varrho \geq 0$ ,
- (c)  $u \leq \varrho y^{\alpha+1}$  holds throughout  $\mathbb{R}_+^{n+1}$ ,
- (d) if in (c) the equality is attained in at least one point of  $\mathbb{R}_+^{n+1}$ , then  $u \equiv \varrho y^{\alpha+1}$ .

By regularization we immediately obtain the following analog for boundary values interpreted in the sense of (1.3).

**Corollary 5.4.** *Let  $\alpha > -1$ . Let  $u$  be a solution to the equation  $D_\alpha u = 0$  in  $\mathbb{R}_+^{n+1}$  such that  $\sup_{S_r} u(x, y) = o(r^{\alpha+1})$  as  $r \rightarrow \infty$  and  $\lim_{y \rightarrow 0} u_y = 0$  in  $\mathcal{S}'$ . Then  $u = 0$  in  $\mathbb{R}_+^{n+1}$ .*

*Proof.* Let  $\psi \in \mathcal{C}_0^\infty$  be a compactly supported test function on  $\mathbb{R}^n$  and consider the regularization

$$u_\psi(x, y) = \int u(x - \eta, y) \psi(\eta) d\eta, \quad y > 0,$$

of  $u$ . Thus  $u_\psi(x, y) = u_y * \psi(x)$  so  $D_\alpha u_\psi = 0$  in  $\mathbb{R}_+^{n+1}$ . If we regard  $u_y$  as a distribution in  $\mathcal{S}'$  arising from the map  $\eta \mapsto u_y(\eta)$ , then  $u_\psi$  can also be regarded as the function  $u_\psi(x, y) = \langle u_y, \psi(x - \cdot) \rangle$  obtained by letting  $u_y$  act on  $\eta \mapsto \psi(x - \eta)$ . Since translation is continuous on  $\mathcal{C}_0^\infty$  it thus follows that

$$\limsup_{(x, y) \rightarrow (x_0, 0)} u_\psi(x, y) = \limsup_{(x, y) \rightarrow (x_0, 0)} \langle u_y, \psi(x - \cdot) \rangle = 0$$

for all  $x_0 \in \mathbb{R}^n$  since  $u_y \rightarrow 0$  in  $\mathcal{S}'$ , see Hörmander [17, Theorem 2.1.8]. Moreover, the growth assumption on  $u$  together with the fact that  $\psi$  is compactly supported implies that  $u_\psi$  satisfies the same growth condition. In fact, by assumption we can for every  $\varepsilon > 0$  find  $r_\varepsilon$  such that

$$(5.4) \quad (\xi^2 + \eta^2)^{-(\alpha+1)/2} u(\xi, \eta) < \frac{\varepsilon}{2^{\alpha+1} \|\psi\|_{L^1}}, \quad \xi^2 + \eta^2 \geq r_\varepsilon.$$

If  $\text{supp } \psi \subset (-R, R)$ , it follows that for any  $r \geq r_\varepsilon + R$  we have  $(x - t)^2 + y^2 \geq r_\varepsilon^2$  for  $(x, y) \in S_r$ . By (5.4) this gives

$$u_\psi(x, y) < \frac{\varepsilon}{2^{\alpha+1} \|\psi\|_{L^1}} \int_{-R}^R ((x - t)^2 + y^2)^{(\alpha+1)/2} |\psi(t)| dt \leq \varepsilon r^{\alpha+1}$$

for  $(x, y) \in S_r$ , so  $\sup_{S_r} u_\psi(x, y) = o(r^{\alpha+1})$  as  $r \rightarrow \infty$ . Hence, Theorem 5.3 implies that  $u_\psi \leq 0$  in  $\mathbb{R}_+^{n+1}$ . A repetition of the arguments applied to  $-u$  shows that  $u_\psi = 0$  in  $\mathbb{R}_+^{n+1}$ . Varying  $\psi \in \mathcal{C}_0^\infty$  we conclude that  $u(x, y) = 0$  for all  $(x, y) \in \mathbb{R}_+^{n+1}$ .  $\square$

Unfortunately, the growth conditions imposed in Corollary 5.4 are not compatible with those satisfied by the Poisson integral  $\mathcal{K}_\alpha[f]$  of elements  $f \in w_\alpha \mathcal{D}'_{L^1}$ ; in fact  $L^1(w_\alpha^{-1}) \subset w_\alpha \mathcal{D}'_{L^1}$  and the order relation (5.2) is sharp for  $f \in L^1(w_\alpha^{-1})$ . Hence, stronger results are needed if we are to conclude that  $u = \mathcal{K}_\alpha[f]$  is the unique solution to the Dirichlet problem (1.3) under appropriate growth constraints. In the harmonic case  $\alpha = 0$ , a uniqueness result for the classical Dirichlet problem with continuous boundary data has been obtained by Siegel and Talvila [27, Theorem 3.1] assuming a growth condition of type (5.2). For distributional boundary data, Alvarez, Guzmán-Partida and Pérez-Esteva [1, Theorem 4.1] provide conditions under which functions harmonic in  $\mathbb{R}_+^{n+1}$  may be represented as Poisson integrals of the data, modulo constant multiples of the nontrivial solution  $(x, y) \mapsto y$ .

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